Surface existence equations and surface synthesis

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Abstract

This paper shows how the Gauss-Mainardi-Codazzi equations can be reinterpreted in terms of variations of curvature with respect to arc-length along lines of curvature. Secondly, it reviews the frame-and-position matching method for surface synthesis. Thirdly, it brings these two ideas together to show that frame and position matching for an infinitesimal patch are exactly equivalent to the Gauss-Mainardi-Codazzi surface existence equations. Finally, conclusions are made concerning this result.

1 Introduction

An introduction to the concepts of differential geometry, and their use in surface construction as used in this paper, can be found in Nutbourne and Martin [5]. In classical differential geometry, the usual method of describing a surface is parametrically, by means of the equation $\mathbf{x} = \mathbf{x}(u, v)$. From such a definition, first and second order derivatives with respect to the parameters can be computed which characterise the surface locally. They are combined to produce the first and second fundamental forms as shown below, where E, F, G constitute the first form and L, M, N the second:

$$E = \mathbf{x}_u.\mathbf{x}_u, \qquad F = \mathbf{x}_u.\mathbf{x}_v, \qquad G = \mathbf{x}_v.\mathbf{x}_v, \tag{1.1}$$

$$L = \mathbf{N}.\mathbf{x}_{uu}, \qquad M = \mathbf{N}.\mathbf{x}_{uv}, \qquad N = \mathbf{N}.\mathbf{x}_{vv}. \tag{1.2}$$

$\mathbf{N}$ is the unit surface normal; subscripts denote differentiation with respect to one of the surface parameters.

A problem extensively studied by classical differential geometers is the one of whether a surface exists for some particular choice of E, F, G, L, M, N as functions of u, v. It has been shown that unless these scalar quantities satisfy three differential equations, called the Gauss-Mainardi-Codazzi equations, a surface does not exist (for example, see Eisenhart [1]). These equations are

$$\begin{aligned} LN - M^2 &= \frac{1}{2}H\left[\frac{\partial}{\partial u}\left(\frac{FE_v}{EH} - \frac{G_u}{H}\right) + \frac{\partial}{\partial v}\left(\frac{2F_u}{H} - \frac{E_v}{H} - \frac{FE_u}{EH}\right),\right] \\ L_v - M_u &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}, \\ M_v - N_u &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}. \end{aligned} \tag{1.3}$$

where $H^2 = EG - F^2$, and the Christoffel symbols Γ^k_{ij} are defined by

$$\begin{aligned} \Gamma^1_{11} &= \frac{GE_u - 2FF_u + FE_v}{2H^2}, & \Gamma^2_{11} &= \frac{2EF_u - EE_v - FE_u}{2H^2}, \\ \Gamma^1_{12} &= \frac{GE_v - FG_u}{2H^2}, & \Gamma^2_{12} &= \frac{EG_u - FE_v}{2H^2}, \\ \Gamma^1_{22} &= \frac{2GF_v - GG_u - FG_v}{2H^2}, & \Gamma^2_{22} &= \frac{EG_v - 2FF_v + FG_u}{2H^2}. \end{aligned} \tag{1.4}$$

The Gauss-Mainardi-Codazzi equations are derived from the compatibility requirements that

$$(\mathbf{x}_{uu})_v = (\mathbf{x}_{uv})_u, \qquad (\mathbf{x}_{uv})_v = (\mathbf{x}_{vv})_u. \tag{1.5}$$

However, in what follows, we will be interested in constructing surfaces expressed not in terms of the fundamental forms, but directly in terms of normal and geodesic curvatures of lines of curvature expressed as functions of arc-length. (Lines of curvature are directions across the surface which are locally extrema of normal curvature.) Thus we need to re-express the Gauss-Mainardi-Codazzi equations in terms of normal and geodesic curvatures—also second order differential quantities, and in terms of arc-lengths rather than u, v parameters. Note that arc-lengths s_1, s_2 along lines of curvature do not form a suitable pair of u, v parameters. Consider the curved surface shown in Figure 1; each of the curves shown is a line of curvature on the surface. It is clear that in general, going a distance s_1 along edge A, then s_2 along edge B, does not arrive at the same point as going s_2 along edge C, then s_1 along edge D. Thus, while choice of a pair of u, v parameters uniquely determines a point on the surface, s_1, s_2 cannot be used in this way.

The expression for the Gauss-Mainardi-Codazzi equations in terms of arc-length along lines of curvature is outlined here; many useful ideas are to be found in Scherrer [6, 7, 8]. Because the lines of curvature are

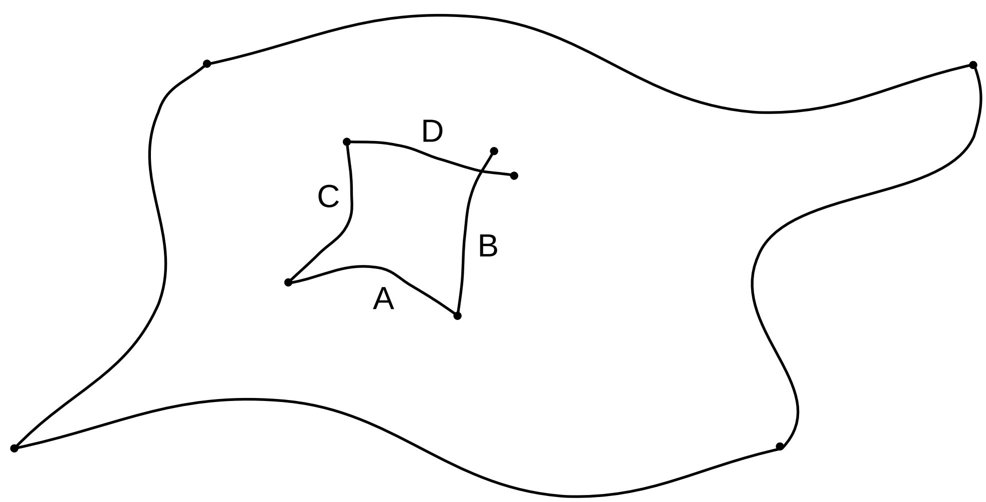

FIGURE 1. Traversing a surface along lines of curvature

orthogonal, the expressions for derivatives with respect to arc-length s_1 and s_2, rather than u, v, for each of the two families are

$$\frac{d}{ds_1} = \frac{1}{\sqrt{E}}\frac{\partial}{\partial u}, \quad \frac{d}{ds_2} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial v}. \tag{1.6}$$

It is also shown in Struik [9] that for orthogonal parametric lines (such as lines of curvature)

$$g_1 = \frac{-E_v}{2E\sqrt{G}}, \quad g_2 = \frac{G_u}{2G\sqrt{E}}, \tag{1.7}$$

where g_1 and g_2 are the geodesic curvatures along the respective lines of curvature.

The desired arc-length form of the Gauss-Mainardi-Codazzi equations can be obtained by substitution of Equations 1.6 and 1.7 into the u, v parametric form of the Gauss-Mainardi-Codazzi equations specialised to lines of curvature (see Struik [9], for example). For this case, $F = 0$ and $M = 0$, and the respective normal curvatures n_1 and n_2 along each of the lines of curvature are

$$n_1 = \frac{L}{E}, \qquad n_2 = \frac{N}{G}. \tag{1.8}$$

The result is the following theorem:

Theorem 1.1 *The Gauss-Mainardi-Codazzi equations may be written in the following form for a curve net whose curves are lines of curvature on a surface. Here, s_i, $i = 1, 2$ measures arc-length along curves of the appropriate family, and g_i, n_i are their geodesic and normal curvatures.*

$$\begin{aligned} n_1 n_2 &= \frac{dg_1}{ds_2} - \frac{dg_2}{ds_1} - g_1^2 - g_2^2, \\ g_1 &= \frac{1}{n_1 - n_2} \frac{dn_1}{ds_2}, \\ g_2 &= \frac{1}{n_1 - n_2} \frac{dn_2}{ds_1}. \end{aligned} \tag{1.9}$$

The first of these results is originally due to Liouville, and may also be found in Scherrer [8]; Struik [9] gives an incorrect version on page 135. A search failed to find the other two results in the literature.

An alternative method of arriving at this Theorem starts by using Equations 1.6 and 1.7 to show that

$$\frac{d}{ds_1}\frac{d}{ds_2} - \frac{d}{ds_2}\frac{d}{ds_1} = -g_1 \frac{d}{ds_1} - g_2 \frac{d}{ds_2}. \tag{1.10}$$

(The fact that the right hand side of this expression is non-zero results from s_1, s_2 not being proper parameters.) Applying this result to the surface frame, using the Bonnet-Kovalevsky Formulae 2.1 explained in the next section, the same expressions for the surface existence relations Equations 1.9 are obtained as before.

2 The frame and position matching method

If we take a surface, then at each point, an orthogonal frame F can be set up, whose basis vectors are the tangent $\mathbf{t}_1$ to a curve 1 of arc length s_1 in the surface, the tangent $\mathbf{t}_2$ to a curve 2 of arc-length s_2 in the surface, orthogonal to $\mathbf{t}_1$, and the unit surface normal $\mathbf{N}$ at that point. It can be shown that on moving across the surface along curve 1, the change in orientation of this frame is given by the Bonnet-Kovalevsky formulae (see Goetz [2])

$$\frac{d}{ds_1}\begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{N} \end{bmatrix} = \begin{bmatrix} 0 & g & n \\ -g & 0 & \tau_g \\ -n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{N} \end{bmatrix}. \tag{2.1}$$

Here, n and g are normal and geodesic curvatures of curve 1, and τ_g is its geodesic torsion.

We can abbreviate this as $dF/ds_1 = \Delta F$ where Δ is the above matrix.

In what follows we will choose curve 1 to be along a line of curvature; because of this $\tau_g = 0$. We also choose curve 2 to be the other line of curvature through the given point; this is compatible, as lines of curvature are always orthogonal. Integrating Equation 2.1 along the arc s_1 gives the new frame orientation at a distant place on the surface. In the special case of a *planar* line of curvature, ϕ, the angle between the curve normal and the surface normal is constant, and a solution can be found in closed form (Martin [3]):

$$\begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{N} \end{bmatrix} (s_1) =$$
$$\begin{bmatrix} \cos\psi & \sin\phi\sin\psi & \cos\phi\sin\psi \\ -\sin\phi\sin\psi & 1-\sin^2\phi\,\mathrm{vers}\,\psi & -\sin\phi\cos\phi\,\mathrm{vers}\,\psi \\ -\cos\phi\sin\psi & -\sin\phi\cos\phi\,\mathrm{vers}\,\psi & 1-\cos^2\phi\,\mathrm{vers}\,\psi \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{N} \end{bmatrix} (0), \quad (2.2)$$

where $\mathrm{vers}\,\psi = 1 - \cos\psi$, $\psi = \int K ds$, $n = K\sin\phi$, $g = K\cos\phi$, and $K^2 = n^2 + g^2$ (K is the plane curvature of the curve).

This can be abbreviated as $F(s_1) = MF(0)$, where M is called the *movement* matrix.

Because $\mathbf{t}_1$ is the tangent to the curve, integrating the first row of Equation 2.1 a second time gives the new position of the frame on the surface.

The previous ideas in this section form the basis of a method of constructing four-sided surface patches, called *principal patches*, where each of the patch boundaries is a line of curvature on the surface (see Martin [3, 4]). Consider a four-sided principal patch $ABCD$ as shown in Figure 2. There are two routes which start from point A and proceed to point C: via sides 1 then 4, or via sides 2 then 3.

Suppose we choose curves for each side of the patch, which have normal and geodesic curvatures n and g defined as functions of arc-length. In order for these curves to form the boundary of a well-defined patch, we must clearly satisfy two conditions:

Position matching Starting from A, the same point must be reached by either route.

Frame matching Starting from A, the same surface orientation resulting at C must be the same by either route.

The motivation for using the ideas above is that we wish to construct surfaces patches whose curvatures we can closely control. Before we can use this method to generate surfaces, however, we need to know whether

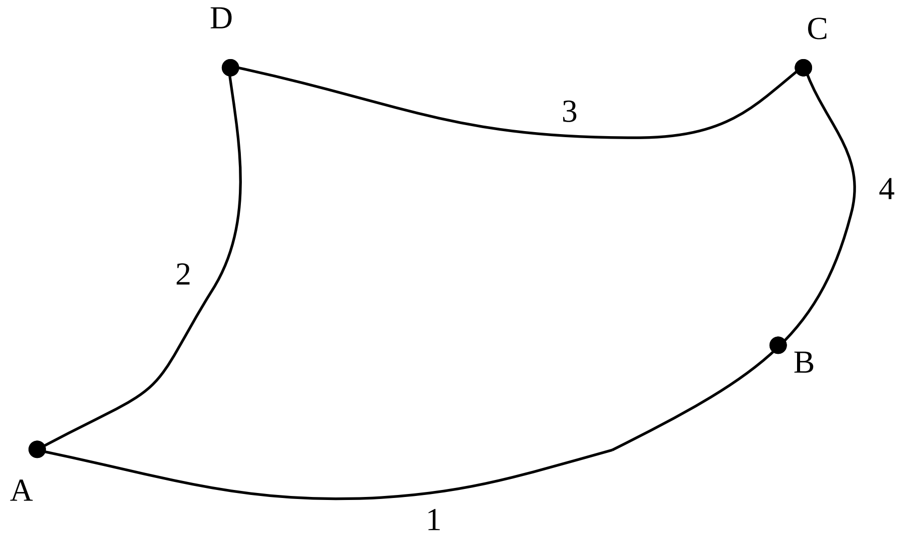

FIGURE 2. Traversing a surface along lines of curvature

these two conditions are sufficient as well as necessary, or whether extra constraints must also be placed on the choices of n and g. To do so, we now proceed to examine frame matching and position matching in detail for infinitesimal patches.

3 Frame matching for an infinitesimal patch

Consider sides $1, 3$ to be s_1-type sides and sides $2, 4$ to be s_2-type sides. The frame matching equation is

$$V_4 M_1 = M_3 V_2 \tag{3.1}$$

where V is the movement matrix for an s_2 type side; single subscripts denote side of the patch. These V matrices are slightly different to the M matrices in that they go along the direction given by $\mathbf{t}_2$ rather than $\mathbf{t}_1$. They are simply related to the M matrices by taking a left turn the start of such a side in order to proceed in the direction of a new $\mathbf{t}_1'$ aligned with the original $\mathbf{t}_2$, then using the usual M matrix, then by taking a right turn at the end of the side:

$$V = RML, \tag{3.2}$$

where R is the right-turn matrix which rotates the frame by $+\pi/2$ about $\mathbf{N}$, and L is its inverse, the left-turn matrix.

Let us now consider an infinitesimal patch. Take a Taylor expansion of the frame orientation to second order using the Bonnet-Kovalevsky formulae 2.1

$$\begin{aligned} F(ds_1) &= F(0) + \frac{dF}{ds_1}(0)\, ds_1 + \frac{1}{2}\frac{d^2F}{ds_1^2}(0)\, ds_1^2 \\ &= F(0) + \Delta F(0)\, ds_1 + \frac{1}{2}\frac{d\Delta F}{ds_1}(0)\, ds_1^2 \qquad (3.3) \\ &= F(0)(I + \Delta\, ds_1 + \frac{1}{2}\frac{d\Delta}{ds_1}\, ds_1^2) + \frac{1}{2}\Delta^2\, ds_1^2 \end{aligned}$$

Using Equation 2.1 to substitute for Δ in the above, we obtain the following expression for movement matrix for an infinitesimal patch (where g_{ij} means dg_i/ds_j etc.):

$$M_1 = \begin{bmatrix} 1 - \frac{1}{2}K_1^2 ds_1^2 & g_1 ds_1 + \frac{1}{2}g_{11} ds_1^2 & n_1 ds_1 + \frac{1}{2}n_{11} ds_1^2 \\ -g_1 ds_1 - \frac{1}{2}g_{11} ds_1^2 & 1 - \frac{1}{2}g_1^2 ds_1^2 & -\frac{1}{2}n_1 g_1 ds_1^2 \\ -n_1 ds_1 - \frac{1}{2}n_{11} ds_1^2 & -\frac{1}{2}n_1 g_1 ds_1^2 & 1 - \frac{1}{2}n_1^2 ds_1^2 \end{bmatrix}. \qquad (3.4)$$

(This formula is correct for all cases, whether the lines of curvature are planar or not.)

Quantities involving sides $3, 4$ can be replaced, to the order required, by

$$\begin{aligned} K_3 &= K_1 + K_{12} ds_2, & K_4 &= K_2 + K_{21} ds_1, \\ n_3 &= n_1 + n_{12} ds_2, & n_4 &= n_2 + n_{21} ds_1, \qquad (3.5) \\ g_3 &= g_1 + g_{12} ds_2, & g_4 &= g_2 + g_{21} ds_1. \end{aligned}$$

Also, we can express ds_3 and ds_4 in terms of ds_1 and ds_2 as

$$\begin{aligned} ds_3 &= ds_1 + \alpha ds_1^2 + \mu ds_1 ds_2 + \lambda ds_2^2, \qquad (3.6) \\ ds_4 &= ds_2 + \gamma ds_1^2 + \delta ds_1 ds_2 + \beta ds_2^2. \end{aligned}$$

However, by considering a patch in which ds_2 tends to zero independently of ds_1, it can be seen that as ds_2 goes to zero, ds_3 must approach ds_1. Thus α, β in Equations 3.6 must be zero.

On equating the components of the frame matching Equation 3.1 up to order ds^2, the only distinct non-trivial relations are the three below:

$$\frac{1}{2}g_{11} ds_1^2 + g_{21} ds_1 ds_2 + (\gamma ds_1^2 + \delta ds_1 ds_2) g_2 + \frac{1}{2}g_{44} ds_2^2 = \qquad (3.7)$$

$$\frac{1}{2}g_{22} ds_2^2 + \frac{1}{2}g_{33} ds_1^2 + g_{12} ds_1 ds_2 + (\mu ds_1 ds_2 + \lambda ds_2^2) g_1 - n_1 n_2 ds_1 ds_2,$$

$$\frac{1}{2}n_{11} ds_1^2 = n_2 g_1 ds_1 ds_2 + n_{12} ds_1 ds_2 + (\mu ds_1 ds_2 + \lambda ds_2^2) n_1 + \frac{1}{2}n_{33} ds_1^2,$$

$$\frac{1}{2}n_{22} ds_2^2 = -n_1 g_2 ds_2 ds_1 + (\gamma ds_1^2 + \delta ds_1 ds_2) n_2 + n_{21} ds_1 ds_2 + \frac{1}{2}n_{44} ds_2^2.$$

To the order required, $g_{33} = g_{11}$ etc., as in the limit $ds_2 \to 0$, sides $1,3$ must be the same. Thus, on separately equating the coefficients of ds_1^2, $ds_1 ds_2$, ds_2^2 in Equations 3.7, we obtain the following relations:

$$\gamma = 0, \qquad \lambda = 0; \tag{3.8}$$

$$\begin{aligned} g_{21} + g_2\delta &= g_{12} + g_1\mu - n_1 n_2, \\ n_2 g_1 + n_{12} + n_1\mu &= 0, \\ -n_1 g_2 + n_2\delta + n_{21} &= 0. \end{aligned} \tag{3.9}$$

4 Position matching for an infinitesimal patch

To first order in ds_1, from Equation 3.4, $\mathbf{t}_1$ is given by

$$\mathbf{t}_1(ds_1) = [1, g_1 ds_1, n_1 ds_1], \tag{4.1}$$

and so to find the chord vector $\mathbf{p}_1$ for side 1, up to second order, this must be integrated:

$$\mathbf{p}_1 = \int_0^{ds_1} \mathbf{t}_1(\sigma)\, d\sigma = \left[ds_1, \frac{1}{2} g_1 ds_1^2, \frac{1}{2} n_1 ds_1^2 \right]. \tag{4.2}$$

The position matching equation can be written as

$$\mathbf{p}_1 + \mathbf{p}_4 L M_1 = \mathbf{p}_2 L + \mathbf{p}_3 V_2, \tag{4.3}$$

where the M_1 and V_2 matrices are needed because the frames at the initial ends of sides $3,4$ have been rotated relative to the reference frame at the original corner, while the L matrices are needed for sides 2 and 4 because these are type s_2 sides.

On expanding Equation 4.3, neglecting terms of higher than second order, the following three equations result:

$$\begin{aligned} ds_1 - \frac{1}{2} g_4 ds_4^2 - g_1 ds_1 ds_4 &= -\frac{1}{2} g_2 ds_2^2 + ds_3, \\ \frac{1}{2} g_1^2 ds_1^2 + ds_4 &= ds_2 + g_2 ds_2 ds_3 + \frac{1}{2} g_3 ds_3, \\ \frac{1}{2} n_1 ds_1^2 + \frac{1}{2} n_4 ds_4^2 &= \frac{1}{2} n_2 ds_2^2 + \frac{1}{2} n_3 ds_3^2. \end{aligned} \tag{4.4}$$

On substituting Equations 3.5 and 3.6 into Equation 4.4 to remove terms involving sides $3,4$, and equating the coefficients of each different term in ds_i and $ds_i ds_j$, $i,j = 1,2$, separately as before, the only distinct results left are

$$\mu = -g_1, \qquad \delta = g_2. \tag{4.5}$$

5 Conclusion

On combining the results of position matching, Equations 4.5, with those of frame matching, Equations 3.9, we obtain the following relationships:

$$\begin{aligned} g_{21} + g_2^2 &= g_{12} - g_1^2 - n_1 n_2, \\ n_2 g_1 + n_{12} - n_1 g_1 &= 0, \\ -n_1 g_2 + n_2 g_2 + n_{21} &= 0. \end{aligned} \tag{5.1}$$

However, these are *exactly* the three surface existence relations expressed in terms of arc-lengths and normal and geodesic curvatures given in Equation 1.9. We may thus state the following theorem:

Theorem 5.1 *Necessary and sufficient conditions for the existence of a principal patch with boundary curves having given functions for geodesic and normal curvatures are the frame and position matching Equations 3.1 and 4.3. Such a patch will necessarily obey the Gauss-Codazzi-Mainardi surface existence equations.*

The most obvious consequence of this result is that no further conditions other than frame and position matching are required on the patch boundary curves to ensure that they are a valid choice for lines of curvature on a proper surface.

Alternatively, we can regard the above result as a geometric interpretation of the meaning of the Gauss-Codazzi-Mainardi surface existence equations, in the case where the parameter lines are lines of curvature:

Theorem 5.2 *The Gauss-Codazzi-Mainardi surface existence equations are equivalent to the observation that, in the case of a four-sided patch bounded by lines of curvature, for a valid surface to exist, the normal and geodesic curvatures of the boundary curves must vary in such a way that frame matching and position matching are satisfied.*

It may be noted in closing that a closed form for frame and position matching may be obtained in the case where all four boundary curves are circular arcs, in which case the surfaces formed are Dupin's cyclides [3, 4, 5].

References

[1] L. P. Eisenhart *A Treatise on the Differential Geometry of Curves and Surfaces* Ginn & Co., 1909.

[2] A. Goetz *Introduction to Differential Geometry* Addison Wesley, 1970.

[3] R. R. Martin *Principal Patches for Computational Geometry* Ph.D. Thesis, Cambridge University Engineering Department, 1982.

[4] R. R. Martin Principal Patches—A New Class of Surface Patch Based on Computational Geometry *Proc. Eurographics '83*, Ed. P. J. W. ten Hagen, 47–56, North-Holland, 1983.

[5] A. W. Nutbourne, R. R. Martin *Differential Geometry Applied to Curve and Surface Design, Vol 1.* Ellis Horwood, 1988.

[6] W. Scherrer Die Grundgleichungen der Flachentheorie 1, *Commentarii Mathematici Helvetici* **29**, 180–199, 1954.

[7] W. Scherrer Die Grundgleichungen der Flachentheorie 2, *Commentarii Mathematici Helvetici* **32**, 73–84, 1957.

[8] W. Scherrer Die Grundgleichungen der Flachentheorie 3, *Commentarii Mathematici Helvetici* **37**, 177–197, 1962.

[9] D. J. Struik *Differential Geometry* Addison Wesley, 1950.