Global Exponential Stability of Bidirectional Associative Memory Neural Networks With Time Delays

Xin-Ge Liu, Ralph R. Martin, Min Wu, and Mei-Lan Tang

Abstract—**In this paper, we consider delayed bidirectional associative memory (BAM) neural networks (NNs) with Lipschitz continuous activation functions. By applying Young's inequality and Hölder's inequality techniques together with the properties of monotonic continuous functions, global exponential stability criteria are established for BAM NNs with time delays. This is done through the use of a new Lyapunov functional and an M-matrix. The results obtained in this paper extend and improve previous results.**

Index Terms—**Bidirectional associative memory (BAM) neural networks (NNs), global exponential stability, Lyapunov functionals, Young's inequality.**

I. INTRODUCTION

NEURAL networks (NNs) have many important applications. Delayed versions of NNs have also proved to be important for solving certain classes of motion-related optimization problems. Various results concerning the dynamical behavior of NNs with delays have been reported in [1]–[9]. Since Kosto [10] introduced bidirectional associative memory (BAM) NNs, researchers have paid particular attention to the stability analysis of BAM NNs with time delays, as such NNs have been shown to be a useful network model for applications in pattern recognition, optimization, and automatic control (see, for example, [11]–[18]). Various sufficient conditions have been presented for the stability of BAM NNs, most of which require that the activation functions are bounded and Lipschitz continuous. Arik [19] studied global asymptotic stability of BAM NNs with time delays, and presented a sufficient condition for uniqueness and global *asymptotic* stability of the equilibrium point. However, his analysis requires that the activation functions and signal propagation functions are bounded and Lipschitz continuous. Furthermore, he did not study global *exponential* stability of BAM NNs.

Global asymptotic stability of a BAM NN only guarantees that the solution of the BAM NN converges to the equilibrium point and says nothing about the rate of convergence of the solution to the equilibrium point. Global exponential stability ensures exponential convergence of the BAM NN to its equilibrium point. In practice, applications of NNs with time delays often require that the network has a unique equilibrium point which is globally exponentially stable, if the network is to be suitable for solving problems in real time. Furthermore, in the case of globally exponentially stable BAM NNs with delays, it is easier to make a quantitative analysis, and thus to determine the convergence behavior of the delayed BAM NN, such as its convergence rate and its precision.

This paper thus considers global exponential stability of BAM NNs with constant time delays. We do *not* assume that the activation functions and signal propagation functions are bounded. Several new sufficient criteria are derived for the existence, uniqueness, and global exponential stability of the equilibrium point of BAM NNs, by constructing a suitable Lyapunov functional, introducing a real parameter, and applying Young's inequality, Hölder's inequality, and the intermediate value theorem of continuous functions. Our new results extend and improve earlier results in [19]–[21].

II. PRELIMINARIES

A. Definitions and Assumptions

The dynamic behavior of a BAM NN with constant time delays is described by the following set of differential equations:

$$\begin{aligned} \dot{u}_i(t) &= -p_i(u_i(t)) + \sum_{j=1}^{m} w_{ij} f_j(z_j(t-\tau_{ij})) + I_i \\ \dot{z}_j(t) &= -q_j(z_j(t)) + \sum_{i=1}^{n} v_{ji} g_i(u_i(t-\sigma_{ji})) + J_j \end{aligned} \quad (1)$$

where $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

The BAM NN model given in (1) can be regarded as an NN with two layers. n is the number of neurons in the first layer and m is the number of neurons in the second layer. $t \geq 0$ stands for time. $u_i(t)$ and $z_j(t)$ represent the activations (i.e., states) of the ith neuron in the first layer and jth neuron in the second layer at time t, respectively. w_{ij} and v_{ji} are the synaptic connection strengths between the neurons in the two layers, while f_j and g_i represent the activation functions of the neurons and the signal propagation functions, respectively, as in [19]. In the first layer, the state $u_i(t)$ of neuron i is dependent on an external

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constant input I_i and other inputs determined by the outputs of the neurons in the second layer via activation functions f_j; in the second layer, the state $z_j(t)$ of neuron j depends on an external constant input J_j and inputs determined by the outputs of the neurons in the first layer via the activation functions g_i. p_i and q_j are differentiable real functions with positive derivatives defining the neuron charging time. $\tau_{ij} \geq 0$ and $\sigma_{ji} \geq 0$ represent constant time delays; $i = 1, \dots, n$ and $j = 1, \dots, m$.

The initial conditions of the BAM NN given in (1) are assumed to be

$$\begin{aligned} u_i(t) &= \phi_i^*(t), \qquad -\tau \leq t \leq 0, \qquad \tau = \max_{i,j}\{\tau_{ij}\} \\ z_j(t) &= \psi_j^*(t), \qquad -\sigma \leq t \leq 0, \qquad \sigma = \max_{i,j}\{\sigma_{ji}\} \end{aligned}$$

in which the initial value functions $\phi_i^*(t)$ and $\psi_j^*(t)$ are continuous functions on $[-\tau, 0]$ and $[-\sigma, 0]$, respectively.

In order to allow comparison of our results with previous results in the literature, we introduce the following assumptions.

Assumption 1: $f_j(\zeta)$ and $g_i(\zeta)$ are bounded real functions, i.e., there exist positive constants L_j and M_i such that for all $\zeta \in R$

$$|f_j(\zeta)| \leq L_j \quad |g_i(\zeta)| \leq M_i.$$

Assumption 2: The activation functions and the signal propagation functions are Lipschitz continuous, i.e., there exist positive constants α_i and β_j such that for all $\zeta_1, \zeta_2 \in R$

$$\begin{aligned} |f_j(\zeta_1) - f_j(\zeta_2)| &\leq \beta_j |\zeta_1 - \zeta_2| \\ |g_i(\zeta_1) - g_i(\zeta_2)| &\leq \alpha_i |\zeta_1 - \zeta_2|. \end{aligned}$$

In this paper, we assume

$$\begin{aligned} a_i &= \inf_{\zeta \in R} p_i'(\zeta) > 0, \qquad p_i(0) = 0 \\ b_j &= \inf_{\zeta \in R} q_j'(\zeta) > 0, \qquad q_j(0) = 0. \end{aligned}$$

Let $C_1 = C([-\tau, 0], R^n)$ be the Banach space of continuous functions which maps $[-\tau, 0]$ into R^n with the topology of uniform convergence. For any $\varphi^* \in C_1$, we define the r-norm of φ^* to be

$$\|\varphi^*\|_r = \sup_{-\tau \leq \theta \leq 0} \left[\sum_{i=1}^{n} |\varphi_i^*(\theta)|^r \right]^{1/r}$$

where $r \geq 1$ is a constant. For simplicity, we usually denote the r-norm $\|\cdot\|_r$ just by $\|\cdot\|$. A similar r-norm can be defined on the Banach space $C_2 = C([-\sigma, 0], R^m)$.

We now explain other basic concepts needed in this paper.

Definition 1: Let $f : R \to R$ be any continuous function. The upper right *Dini-derivative* $D^+ f$ is defined to be

$$D^+ f(t) = \limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h}.$$

Definition 2: Let Q be a real $n \times n$ matrix with elements Q_{ij} satisfying $Q_{ii} > 0, i = 1, \dots, n$, and $Q_{ij} \leq 0$ for $i \neq j$. Then, Q is said to be a *nonsingular M-matrix* if the real part of every eigenvalue of Q is positive.

Definition 3: Suppose we are given the BAM NN in (1). A vector $(u^*, z^*) = (u_1^*, \dots, u_n^*, z_1^*, \dots, z_m^*)$ is said to be an *equilibrium point* if it satisfies

$$\begin{aligned} p_i(u_i^*) &= \sum_{j=1}^{m} w_{ij} f_j(z_j^*) + I_i, \qquad i = 1, \dots, n \\ q_j(z_j^*) &= \sum_{i=1}^{n} v_{ji} g_i(u_i^*) + J_j, \qquad j = 1, \dots, m. \end{aligned} \tag{2}$$

Definition 4: The aforementioned equilibrium point is said to be *globally exponentially stable* if we can find an $r \geq 1$ such that there exist constants $\varepsilon > 0$ and $M \geq 1$ such that for any $t \geq 0$

$$\left[\sum_{i=1}^{n} |u_i(t) - u_i^*|^r + \sum_{j=1}^{m} |z_j(t) - z_j^*|^r \right]^{1/r} \leq M(\|\phi^* - u^*\| + \|\psi^* - z^*\|) e^{-\varepsilon t}.$$

Definition 5: A map $H : R^n \to R^n$ is a homeomorphism of R^n onto itself, if $H \in C^0$, H is one-to-one, H is onto, and is the inverse map $H^{-1} \in C^0$, where C^0 represents the set of all continuous functions from R^n to R^n.

B. Basic Lemmas

In this section, we start with some lemmas related to nonsingular M-matrices. Let Q be an $n \times n$ matrix with nonpositive off-diagonal element. Q is a nonsingular M-matrix in only the following cases [22].

Lemma 1: Q is a nonsingular M-matrix if there exists a positive vector $\xi > 0$ (i.e., with every element positive) such that $Q\xi > 0$.

Lemma 2: Q is a nonsingular M-matrix if there exists a positive diagonal matrix D, i.e., with each $D_{ii} > 0$, such that QD is strictly row diagonally dominant, i.e.,

$$Q_{ii} D_{ii} > \sum_{j=1, j \neq i}^{n} D_{jj} |Q_{ij}|, \qquad i = 1, \dots, n.$$

Lemma 3: Q is a nonsingular M-matrix if there exists a positive diagonal matrix D such that $Q^T D$ is strictly row diagonally dominant.

If we set D to a unit matrix in Lemmas 2 and 3, we readily obtain Fact 1.

Fact 1: If Q is strictly row (or column) diagonally dominant, then Q is a nonsingular M-matrix.

To prove our main result, the next four lemmas are needed.

Lemma 4: If $H(x) \in C^0$ and satisfies the following conditions:

1) $H(x)$ is injective on R^n;
2) $\lim_{\|x\| \to +\infty} \|H(x)\| = +\infty$;

then $H(x)$ is a homeomorphism of R^n [23].

Lemma 5: Assume that $a > 0$, $b > 0$, $p > 1$, and $1/p + 1/q = 1$. Then, the following inequality, *Young's inequality* [12], holds:

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Clearly, if $a = 0$ or $b = 0$, this extended version of Young's inequality still holds.

Lemma 6: If $p > 1$ and $1/p + 1/q = 1$, then the following inequality, *Hölder's inequality*, holds:

$$\sum_{j=1}^{n} |a_j b_j| \le \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q\right)^{1/q}.$$

Lemma 7: Assume that $t > 0, \delta > 0, \theta > 0$, and $f : R \to R$ is a continuous function. Then, the following inequality holds:

$$\int_0^t e^{\theta s} ds \int_{s-\delta}^{s} |f(v)|^r dv \le \delta \int_{-\delta}^{t} |f(v)|^r e^{\theta(v+\delta)} dv \tag{3}$$

where r is a positive constant.

Proof: Let $D_1 = \{0 \le s \le t, s - \delta \le v \le s\}$ and $D_2 = \{-\delta \le v \le t, v \le s \le v + \delta\}$. No matter whether $0 < t < \delta$ or $t \ge \delta$, we have $D_1 \subseteq D_2$. So

$$\begin{aligned}\int_0^t e^{\theta s} ds \int_{s-\delta}^{s} |f(v)|^r dv &\le \int_{-\delta}^{t} dv \int_{v}^{v+\delta} |f(v)|^r e^{\theta s} ds \\ &\le \int_{-\delta}^{t} dv \int_{v}^{v+\delta} |f(v)|^r e^{\theta(v+\delta)} ds \\ &= \delta \int_{-\delta}^{t} |f(v)|^r e^{\theta(v+\delta)} dv. \end{aligned} \tag{4}$$

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III. GLOBAL STABILITY ANALYSIS

In this section, we present a new sufficient condition which guarantees the existence, uniqueness, and global exponential stability of the equilibrium point of the system given in (1).

Throughout this paper, we use the notation $\alpha^{-r} = \operatorname{diag}(\alpha_1^{-r}, \ldots, \alpha_n^{-r})$, $\beta^{-r} = \operatorname{diag}(\beta_1^{-r}, \ldots, \beta_m^{-r})$, $A^r = \operatorname{diag}(a_1^r, \ldots, a_n^r)$, and $B^r = \operatorname{diag}(b_1^r, \ldots, b_m^r)$; W^r is an $m \times n$ matrix with $W_{ji}^r = |w_{ij}|^r$ and V^r is an $n \times m$ matrix with $V_{ij}^r = |v_{ji}|^r$, where $r \ge 1$, $i = 1, \ldots, n$, and $j = 1, \ldots, m$. We write

$$Q = \begin{bmatrix} nm^{-r}B^r\beta^{-r} & -W^r \\ -V^r & mn^{-r}A^r\alpha^{-r} \end{bmatrix}. \tag{5}$$

Theorem 1: The BAM NN given in (1), under Assumption 2, has a unique equilibrium point which is globally exponentially stable if there exists a number $r \ge 1$ such that Q is a nonsingular M-matrix (using the same notation as before).

The proof of Theorem 1 is given in the Appendix.

Since global exponential stability implies global asymptotic stability, we can obtain the following results.

Corollary 1: The BAM NN given in (1), under Assumption 2, has a unique equilibrium point which is globally asymptotically stable if there exists a number $r \ge 1$ such that Q is a nonsingular M-matrix (again, using the same notation).

Remark 1: If we set $r = 2$, $p_i(u_i(t)) = a_i u_i(t)$, and $q_j(z_j(t)) = b_j z_j(t)$ in Corollary 1, Corollary 1 becomes the same as [19, Th. 1]. However, using our methods, Corollary 1 does not require Assumption 1 to hold. Thus, [19, Th. 1] is a special case of Corollary 1 here. Furthermore, in this setting, Theorem 1 in our paper shows that the BAM NN is globally exponentially stable if it satisfies the same conditions as for [19, Th. 1]. Therefore, Theorem 1 in our paper is a stronger version of the result in [19].

Since strictly row (or column) diagonally dominant matrices are nonsingular M-matrices by Fact 1, it is easy for us to obtain the following further corollary.

Corollary 2: The BAM NN given in (1), under Assumption 2, has a unique equilibrium point which is globally exponentially stable if there exists a number $r \ge 1$ such that Q is strictly row (or column) diagonally dominant (again, using the same notation).

The result of Corollary 2 is important as it allows us to impose constraint conditions on matrices W^r and V^r independently of each other, whereas checking whether Q in Theorem 1 is a nonsingular M-matrix requires the establishment of a relationship between the elements of matrices W^r and V^r.

IV. COMPARISONS AND EXAMPLES

In this section, we compare our results with previous results in the literature, which are restated in the following.

Theorem 2 (From [20]): Under Assumptions 1 and 2, the equilibrium point of the NN defined by (1) with $p_i(u_i(t)) = a_i u_i(t)$ and $q_j(z_j(t)) = b_j z_j(t)$ is globally exponentially stable if there exist constants $\lambda_i > 0$ and $\mu_j > 0$ such that

$$\begin{aligned}\sum_{j=1}^{m} \lambda_i \beta_j |w_{ij}| + \sum_{j=1}^{m} \mu_j \alpha_i |v_{ji}| &< 2a_i\lambda_i \\ \sum_{i=1}^{n} \lambda_i \beta_j |w_{ij}| + \sum_{i=1}^{n} \mu_j \alpha_i |v_{ji}| &< 2b_j\lambda_j.\end{aligned}$$

Theorem 3 (From [21]): Under Assumptions 1 and 2, the equilibrium point of the NN defined by (1) with $p_i(u_i(t)) = a_i u_i(t)$ and $q_j(z_j(t)) = b_j z_j(t)$ is globally exponentially stable if the following conditions hold:

$$\begin{aligned}\sum_{j=1}^{m} \beta_j |w_{ij}| &< a_i \\ \sum_{i=1}^{n} \alpha_i |v_{ji}| &< b_j.\end{aligned}$$

Theorems 1–3 provide different sufficient conditions for the global exponential stability of the BAM NN in (1) even if we set $p_i(u_i(t)) = a_i u_i(t)$ and $q_j(z_j(t)) = b_j z_j(t)$. Note that the activation functions and signal propagation functions in Theorems

2 and 3 are required to be bounded and Lipschitz continuous. However, Theorem 1 only requires the activation functions and signal propagation functions to satisfy the Lipschitz condition. The assumption in Theorem 1 is weaker than those in Theorems 2 and 3.

To further illustrate the differences between these three criteria, we analyze the stability of the BAM NN given in Example 1 using these three criteria and compare the ranges of permissible synaptic connection strengths w_{ij} and v_{ji} when we use these three criteria to design a BAM NN which is globally exponentially stable.

Example 1: Consider the following first-order BAM NN:

$$\begin{aligned}\dot{u}(t) &= -u(t) + wg(z(t-\tau)) + I\\ \dot{z}(t) &= -z(t) + vg(u(t-\sigma)) + J\end{aligned}$$

with $|g(\zeta_1) - g(\zeta_2)| \leq |\zeta_1 - \zeta_2|$ for all $\zeta_1, \zeta_2 \in R$.

Theorems 2 and 3 cited previously and [19, Th. 1] are not directly applicable to this example unless the function g is bounded or the equilibrium point exists.

Arik [19] only shows that this BAM NN is globally asymptotically stable when $|w||v| < 1$ under the strict assumption that g is bounded. The criterion for global asymptotic stability derived by Arik [19] cannot be applied to analyze the global exponential stability of this BAM NN. However, using our Theorem 1, we find that this BAM NN is globally exponentially stable when $|w||v| < 1$ for any $r \geq 1$ even if the function g is unbounded. Since the boundedness requirements of the activation functions and signal propagation functions are removed, Theorem 1 is not as restrictive as that in [19]. Since global exponential stability implies the global asymptotic stability, this BAM NN is also globally asymptotically stable when the function g is unbounded. Therefore, our result improves the criterion for globally asymptotical stability derived by Arik [19] for this particular example.

Using [20, Th. 2] and Lemma 1, the equilibrium point of this first-order BAM NN is globally exponentially stable if $|w| + |v| < 2$. Now, if $|w| + |v| < 2$, then $2\sqrt{|w||v|} \leq |w| + |v| < 2$, i.e., $|w||v| < 1$. The condition $|w||v| < 1$ is weaker than the condition $|w| + |v| < 2$. Note that Theorems 1 and 2 provide different sufficient conditions. For global exponential stability of this first-order BAM NN, [21, Th. 3] requires that $|w| < 1$ and $|v| < 1$. Obviously, the condition $|w||v| < 1$ is weaker than the conditions $|w| < 1$ and $|v| < 1$. Again, note that Theorems 1 and 3 give different sufficient conditions.

Next, we consider the stability and numerical simulation of the following BAM NN with unbounded activation functions and signal propagation functions.

Example 2: Consider the following BAM NN:

$$\begin{aligned}\dot{u}_i(t) &= -p_i(u_i(t)) + \sum_{j=1}^{2} w_{ij} f_j(z_j(t-\tau_{ij})) + I_i\\ \dot{z}_j(t) &= -q_j(z_j(t)) + \sum_{i=1}^{2} v_{ji} g_i(u_i(t-\sigma_{ji})) + J_j\end{aligned}$$

where $i = 1, 2$, $j = 1, 2$, and the activation functions f_j and signal propagation functions g_i are unbounded and satisfy Lipschitz conditions

$$\begin{aligned}|f_1(\zeta_1) - f_1(\zeta_2)| &\leq |\zeta_1 - \zeta_2|\\ |f_2(\zeta_1) - f_2(\zeta_2)| &\leq 4|\zeta_1 - \zeta_2|\\ |g_1(\zeta_1) - g_1(\zeta_2)| &\leq 2|\zeta_1 - \zeta_2|\\ |g_2(\zeta_1) - g_2(\zeta_2)| &\leq 3|\zeta_1 - \zeta_2|\end{aligned}$$

$$\begin{aligned}w_{11} &= 0.4673 \quad w_{12} = 0.100 \quad w_{21} = 0.20\\ v_{11} &= 0.5170 \quad v_{12} = 0.210 \quad v_{21} = 0.15\\ w_{22} &= 1.0084 \quad v_{22} = 1.203\end{aligned}$$

and

$$\begin{aligned}p_1(\zeta) &= 5\zeta + 0.5\sin\zeta \quad p_2(\zeta) = 5\zeta + 0.25\sin\zeta\\ q_1(\zeta) &= 5\zeta + 0.2\cos\zeta \quad q_2(\zeta) = 5\zeta + 0.4\cos\zeta.\end{aligned}$$

Since the activation functions f_j and signal propagation functions g_i in this BAM NN are unbounded, the conditions in Theorems 2 and 3 cannot be satisfied; so, these two global exponential stability criteria cannot be applied to analyze the global exponential stability of this BAM NN. Theorem 1 in [19] also cannot be applied to the global asymptotic stability analysis of this BAM NN. The Lipschitz constants for activation functions f_j and signal propagation functions g_i can be determined to be $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = 1$, and $\beta_2 = 4$. Since $p_i(\zeta)$ and $q_j(\zeta)$ are differentiable functions, we have $p_1'(\zeta) \geq a_1 = 4.5$, $p_2'(\zeta) \geq a_2 = 4.75$, $q_1'(\zeta) \geq b_1 = 4.8$, and $q_2'(\zeta) \geq b_2 = 4.6$. Using Theorem 1, we may set $r = 1.5$ such that the following matrix:

$$Q = \begin{bmatrix} 7.4361 & 0 & -0.3194 & -0.0894\\ 0 & 55.8100 & -0.0316 & -1.0126\\ -0.3717 & -0.0581 & 19.0919 & 0\\ -0.0962 & -1.3195 & 0 & 38.0371 \end{bmatrix}$$

is a nonsingular M-matrix. In fact, the eigenvalues of the matrix Q in Theorem 1 are 7.4257, 19.1020, 37.9625, and 55.8849—they are all positive. Therefore, the BAM NN given in Example 2 has a unique equilibrium point which is globally exponentially stable.

For numerical simulation, let $I_1 = I_2 = J_1 = J_2 = 0.00001$ and the delay parameter $\tau = \sigma = 0.2$. The initial conditions of the BAM NN given in Example 2 are assumed to be $\phi_i^*(t) = 0$ and $\psi_j^*(t) = 0$ for $-0.2 \leq t \leq 0$. Fig. 1 depicts the time responses of state variables with step $h = 0.01$. It confirms that the proposed criterion leads to the unique and globally exponentially stable equilibrium point $(u_1^*, u_2^*, z_1^*, z_2^*) = (-0.0758, -0.6331, -0.1686, -0.6561)$.

Remark 2: The BAM NN model, an extension of the unidirectional autoassociator of Hopfield [24], was first introduced by Kosko [10]. Although the BAM NN model given in (1) can be mathematically regarded as a Hopfield-type NN with dimension $n + m$ or a Cohen–Grossberg NN with dimension $n + m$, it has many special properties due to the particular structure of the connection weights, and has practical applications in storing

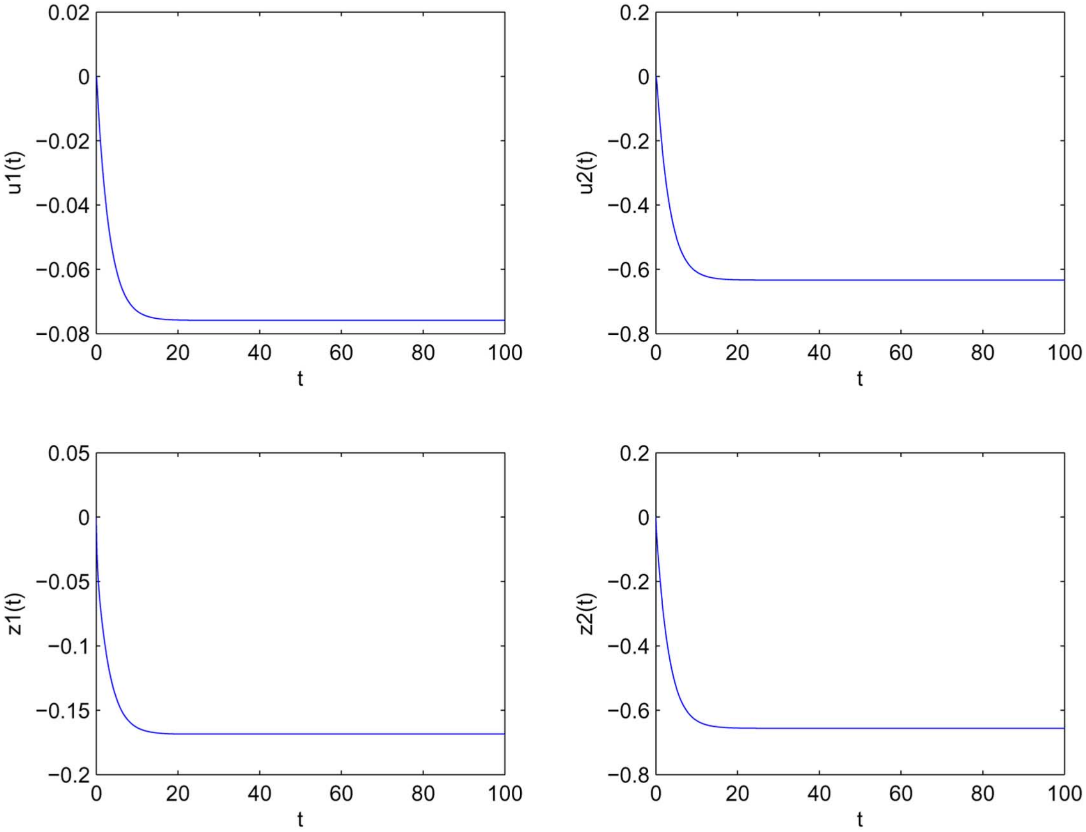

Fig. 1. Transient response of state variable u_i and z_j.

paired patterns or memories. By constructing proper vector Lyapunov functions, using M-matrix theory and qualitative property of the differential inequalities, Zhang [25] established the sufficient conditions for global exponential stability of the equilibrium point for Cohen–Grossberg NNs in which the activation functions are unbounded. However, when $r > 1$, Theorem 1 is different from the result obtained by combining [25, Th. 2] with [25, Th. 1]: Theorem 1 is not a corollary of [25, Th. 1 and 2].

Remark 3: Using fixed-point theorem in Banach space and differential inequality techniques, Zhao [26] obtained new sufficient conditions ensuring the global exponential stability and existence of periodic solutions for cellular NNs with variable delays. By employing homeomorphism theory and the inequality $a\sum_{k=1}^{m} b_k^{q_k} \le (1/r)\sum_{k=1}^{m} q_k b_k^r + (1/r)a^r$ with $a \ge 0, b_k \ge 0$, $q_k > 0$, and $\sum_{k=1}^{m} q_k = r-1$, Zhao and Cao [27] derived some important conditions ensuring the existence and uniqueness of the equilibrium point, and its global exponential stability, for cellular NNs with constant delays and without assuming boundedness for the signal functions. Furthermore, using the same inequality in [27] and further ideas, Zhao [28] presented other new criteria ensuring global exponential stability and existence of periodic oscillatory solutions of BAM NN with constant delays. By introducing ingenious real parameters, employing suitable Lyapunov functionals, and applying contracting mapping, Zhao and Wang [29] gave a set of novel sufficient conditions ensuring the existence, uniqueness, and global exponential stability of periodic oscillatory solutions of a class of reaction–diffusion NNs with constant delays and time-varying coefficients. In this paper, however, we have only investigated the existence, uniqueness of equilibrium point, and global exponential stability of BAM NNs using an M-matrix, Young's inequality, and Hölder's inequality, instead of the inequality used in [27]. The sufficient conditions obtained in this paper are different from those in [27].

V. CONCLUSION

The main contribution of this paper is a result that ensures the existence, uniqueness, and global exponential stability of the equilibrium point of BAM NNs with time delays, without assuming that the activation functions and signal propagation functions are bounded. New sufficient conditions for ascertaining global exponential stability, i.e., Theorem 1 and Corollary 2, have been derived for BAM NNs, improving and extending previous work.

APPENDIX
PROOF OF THEOREM 1

We here prove Theorem 1.

Proof: We consider here the case when $r > 1$, using a sequence of three steps. The case $r = 1$ can be directly proved in three similar steps, but in which we use a simple computation (omitted here) instead of Young's inequality.

If (u^*, z^*) is an equilibrium point of the BAM NN given in (1), then (u^*, z^*) satisfies (2).

Let

$$H(u, z) : R^{n+m} \to R^{n+m} \tag{6}$$

where

$$H(u,z)=(H_1(u,z),\ldots,H_{n+m}(u,z))$$

with

$$\begin{aligned} H_i(u,z)&=-p_i(u_i)+\sum_{j=1}^{m}w_{ij}f_j(z_j)+I_i\\ H_{n+j}(u,z)&=-q_j(z_j)+\sum_{i=1}^{n}v_{ji}g_i(u_i)+J_j \end{aligned}$$

where $i=1,\ldots,n$ and $j=1,\ldots,m$.

Obviously, the solution of $H(u,z)=0$ is the equilibrium point of the BAM NN given in (1). Hence, the existence and the uniqueness of the equilibrium of the BAM NN given in (1) can be reduced to proving that the map $H(u,z)$ is a homeomorphism of R^{n+m}.

Step 1) We prove that $H(u,z)$ is an injective map on R^{n+m} by showing that assuming otherwise leads to a contradiction.

We suppose that values (u,z) and $(\bar{u},\bar{z})$ exist in R^{n+m} such that $(u,z)\neq(\bar{u},\bar{z})$ while $H(u,z)=H(\bar{u},\bar{z})$, so $u-\bar{u}=(u_1-\bar{u}_1,\ldots,u_n-\bar{u}_n)\neq 0$ or $z-\bar{z}=(z_1-\bar{z}_1,\ldots,z_m-\bar{z}_m)\neq 0$.

Since Q is a nonsingular M-matrix, by Lemma 1, then there exists a positive vector $\Psi=(\mu_1,\ldots,\mu_m,\lambda_1,\ldots,\lambda_n)^T$ such that $Q\Psi>0$, so

$$\begin{aligned} \Omega_j&=nm^{-r}\mu_jb_j^r\beta_j^{-r}-\sum_{i=1}^{n}\lambda_i|w_{ij}|^r>0\\ \delta_i&=mn^{-r}\lambda_ia_i^r\alpha_i^{-r}-\sum_{j=1}^{m}\mu_j|v_{ji}|^r>0 \end{aligned} \tag{7}$$

where $i=1,\ldots,n$ and $j=1,\ldots,m$.

Since $\Omega_j>0$, $\delta_i>0$, $r>1$, $\alpha_i>0$, $\beta_j>0$, and $u-\bar{u}\neq 0$ or $z-\bar{z}\neq 0$, we can see that

$$-\sum_{i=1}^{n}n^r\alpha_i^r\delta_i|u_i-\bar{u}_i|^r-\sum_{j=1}^{m}m^r\beta_j^r\Omega_j|z_j-\bar{z}_j|^r<0. \tag{8}$$

Since $H(u,z)=H(\bar{u},\bar{z})$ and p_i and q_j are differentiable functions, there exist $\xi_i\in R$ and $\eta_j\in R$ such that

$$\begin{aligned} -p_i'(\xi_i)(u_i-\bar{u}_i)+\sum_{j=1}^{m}w_{ij}[f_j(z_j)-f_j(\bar{z}_j)]&=0\\ -q_j'(\eta_j)(z_j-\bar{z}_j)+\sum_{i=1}^{n}v_{ji}[g_i(u_i)-g_i(\bar{u}_i)]&=0. \end{aligned}$$

Furthermore

$$\sum_{i=1}^{n}rm\lambda_ia_i^{r-1}|u_i-\bar{u}_i|^{r-1}\mathrm{sign}(u_i-\bar{u}_i)\times\left\{-p_i'(\xi_i)(u_i-\bar{u}_i)+\sum_{j=1}^{m}w_{ij}[f_j(z_j)-f_j(\bar{z}_j)]\right\}=0$$

$$\sum_{j=1}^{m}rn\mu_jb_j^{r-1}|z_j-\bar{z}_j|^{r-1}\mathrm{sign}(z_j-\bar{z}_j)\times\left\{-q_j'(\eta_j)(z_j-\bar{z}_j)+\sum_{i=1}^{n}v_{ji}[g_i(u_i)-g_i(\bar{u}_i)]\right\}=0.$$

As $a_i=\inf_{\zeta\in R}p_i'(\zeta)>0$ and $b_j=\inf_{\zeta\in R}q_j'(\zeta)>0$

$$\begin{aligned} \rho=&\sum_{i=1}^{n}\Bigg\{-rm\lambda_ia_i^r|u_i-\bar{u}_i|^r\\ &\quad+\sum_{j=1}^{m}\Bigg[rm\lambda_ia_i^{r-1}|u_i-\bar{u}_i|^{r-1}|w_{ij}|\\ &\qquad\times|f_j(z_j)-f_j(\bar{z}_j)|\Bigg]\Bigg\}\\ &+\sum_{j=1}^{m}\Bigg\{-rn\mu_jb_j^r|z_j-\bar{z}_j|^r\\ &\quad+\sum_{i=1}^{n}\Bigg[rn\mu_jb_j^{r-1}|z_j-\bar{z}_j|^{r-1}|v_{ji}|\\ &\qquad\times|g_i(u_i)-g_i(\bar{u}_i)|\Bigg]\Bigg\}\\ \geq&\,0. \end{aligned} \tag{9}$$

Next, we bound the value of the left-hand side of (9). By Assumption 2, using the extended version of Young's inequality, we obtain

$$\begin{aligned} \rho\leq&\sum_{i=1}^{n}\Bigg\{-rm\lambda_ia_i^r|u_i-\bar{u}_i|^r\\ &\quad+\sum_{j=1}^{m}r\lambda_i\Bigg[\frac{r-1}{r}\left(a_i^{r-1}|u_i-\bar{u}_i|^{r-1}\right)^{r/r-1}\\ &\qquad+\frac{1}{r}(m|w_{ij}||f_j(z_j)-f_j(\bar{z}_j)|)^r\Bigg]\Bigg\}\\ &+\sum_{j=1}^{m}\Bigg\{-rn\mu_jb_j^r|z_j-\bar{z}_j|^r\\ &\quad+\sum_{i=1}^{n}r\mu_j\Bigg[\frac{r-1}{r}\left(b_j^{r-1}|z_j-\bar{z}_j|^{r-1}\right)^{r/r-1}\\ &\qquad+\frac{1}{r}(n|v_{ji}||g_i(u_i)-g_i(\bar{u}_i)|)^r\Bigg]\Bigg\}\\ \leq&-\sum_{i=1}^{n}n^r\alpha_i^r\delta_i|u_i-\bar{u}_i|^r\\ &-\sum_{j=1}^{m}m^r\beta_j^r\Omega_j|z_j-\bar{z}_j|^r. \end{aligned} \tag{10}$$

From (9) and (10), we find that

$$-\sum_{i=1}^{n}n^r\alpha_i^r\delta_i|u_i-\bar{u}_i|^r-\sum_{j=1}^{m}m^r\beta_j^r\Omega_j|z_j-\bar{z}_j|^r\geq 0 \tag{11}$$

which contradicts (8), and hence implies that $H(u,z)$ is an injective map on R^{n+m}.

Step 2) We prove that

$$\lim_{\|(u,z)\|\to+\infty}\|H(u,z)\|=+\infty.$$

Since $w_{ij}, v_{ji}, f_j(0), g_i(0), I_i$, and J_j are constants, it suffices to show that

$$\lim_{\|(u,z)\|\to+\infty}\|\tilde{H}(u,z)\|=+\infty$$

where $\tilde{H}(u,z) = (\tilde{H}_1(u,z),\ldots,\tilde{H}_{n+m}(u,z)) : R^{n+m}\to R^{n+m}$ and

$$\begin{aligned}\tilde{H}_i(u,z) &= -\,p_i(u_i)+\sum_{j=1}^{m}w_{ij}[f_j(z_j)-f_j(0)]\\ \tilde{H}_{n+j}(u,z) &= -\,q_j(z_j)+\sum_{i=1}^{n}v_{ji}[g_i(u_i)-g_i(0)].\end{aligned}$$

In fact, there exist $\xi_i^*\in R$ and $\eta_j^*\in R$ such that

$$\begin{aligned}&\sum_{i=1}^{n}rm\lambda_i a_i^{r-1}|u_i|^{r-1}\mathrm{sign}(u_i)\tilde{H}_i(u,z)\\ &\quad+\sum_{j=1}^{m}rn\mu_j b_j^{r-1}|z_j|^{r-1}\mathrm{sign}(z_j)\tilde{H}_{n+j}(u,z)\\ &\quad=\sum_{i=1}^{n}rm\lambda_i a_i^{r-1}|u_i|^{r-1}\mathrm{sign}(u_i)\\ &\quad\times\left[-p_i'(\xi_i^*)u_i+\sum_{j=1}^{m}w_{ij}[f_j(z_j)-f_j(0)]\right]\\ &\quad+\sum_{j=1}^{m}rn\mu_j b_j^{r-1}|z_j|^{r-1}\mathrm{sign}(z_j)\\ &\quad\times\left[-q_j'(\eta_j^*)z_j+\sum_{i=1}^{n}v_{ji}[g_i(u_i)-g_i(0)]\right]\\ &\quad\le\sum_{i=1}^{n}\Bigg\{-m\lambda_i a_i^r|u_i|^r+\sum_{j=1}^{m}\Bigg[-(r-1)\lambda_i a_i^r|u_i|^r\\ &\qquad+r\lambda_i\Bigg[\frac{r-1}{r}\left(a_i^{r-1}|u_i|^{r-1}\right)^{r/r-1}\\ &\qquad+\frac{1}{r}(m|w_{ij}||f_j(z_j)-f_j(0)|)^r\Bigg]\Bigg]\Bigg\}\\ &\quad+\sum_{j=1}^{m}\Bigg\{-n\mu_j b_j^r|z_j|^r+\sum_{i=1}^{n}\Bigg[-(r-1)\mu_j b_j^r|z_j|^r\\ &\qquad+r\mu_j\Bigg[\frac{r-1}{r}\left(b_j^{r-1}|z_j|^{r-1}\right)^{r/r-1}\\ &\qquad+\frac{1}{r}(n|v_{ji}||g_i(u_i)-g_i(0)|)^r\Bigg]\Bigg]\Bigg\}\\ &\quad\le-\sum_{i=1}^{n}n^r\alpha_i^r\delta_i|u_i|^r-\sum_{j=1}^{m}m^r\beta_j^r\Omega_j|z_j|^r\\ &\quad\le-\varepsilon\left[\sum_{i=1}^{n}|u_i|^r+\sum_{j=1}^{m}|z_j|^r\right]\end{aligned}\tag{12}$$

where

$$\varepsilon=\min\left\{\min_{1\le i\le n}\left\{n^r\alpha_i^r\delta_i\right\},\min_{1\le j\le m}\left\{m^r\beta_j^r\Omega_j\right\}\right\}.$$

Clearly, $\varepsilon>0$. Thus

$$\begin{aligned}&\sum_{i=1}^{n}rm\lambda_i a_i^{r-1}|u_i|^{r-1}|\tilde{H}_i(u,z)|\\ &\quad+\sum_{j=1}^{m}rn\mu_j b_j^{r-1}|z_j|^{r-1}|\tilde{H}_{n+j}(u,z)|\\ &\quad\ge\varepsilon\left[\sum_{i=1}^{n}|u_i|^r+\sum_{j=1}^{m}|z_j|^r\right].\end{aligned}\tag{13}$$

Let

$$\kappa=\max\left\{\max_{1\le i\le n}\left\{rm\lambda_i a_i^{r-1}\right\},\max_{1\le j\le m}\left\{rn\mu_j b_j^{r-1}\right\}\right\}.$$

Using Hölder's inequality gives

$$\begin{aligned}&\sum_{i=1}^{n}rm\lambda_i a_i^{r-1}|u_i|^{r-1}|\tilde{H}_i(u,z)|\\ &\quad+\sum_{j=1}^{m}rn\mu_j b_j^{r-1}|z_j|^{r-1}|\tilde{H}_{n+j}(u,z)|\\ &\quad\le\kappa\Bigg\{\left(\sum_{i=1}^{n}|u_i|^r+\sum_{j=1}^{m}|z_j|^r\right)^{(r-1)/r}\\ &\qquad\times\left(\sum_{i=1}^{n}|\tilde{H}_i(u,z)|^r+\sum_{j=1}^{m}|\tilde{H}_{n+j}(u,z)|^r\right)^{1/r}\Bigg\}\\ &\quad\le\kappa\Bigg\{\left(\sum_{i=1}^{n}|u_i|^r+\sum_{j=1}^{m}|z_j|^r\right)^{(r-1)/r}\|\tilde{H}(u,z)\|\Bigg\}.\end{aligned}\tag{14}$$

Combining (13) and (14) gives

$$\|\tilde{H}(u,z)\|\ge\frac{\varepsilon}{\kappa}\left(\sum_{i=1}^{n}|u_i|^r+\sum_{j=1}^{m}|z_j|^r\right)^{1/r}$$

so

$$\lim_{\|(u,z)\|\to+\infty}\|\tilde{H}(u,z)\|=+\infty$$

and, therefore

$$\lim_{\|(u,z)\|\to+\infty}\|H(u,z)\|=+\infty.$$

From Steps 1) and 2) and Lemma 4, the map $H(u,z)$ is a homeomorphism of R^{n+m}. Thus, the BAM NN given in (1) has a unique equilibrium point. Let us denote the unique equilibrium point by (u^*,z^*).

Step 3) We now prove that this unique equilibrium point (u^*,z^*) is globally exponentially stable.

In order to simplify our proof, using translation $x_i(t)=u_i(t)-u_i^*$ and $y_j(t)=z_j(t)-z_j^*$, we transform the system given in (1) to the following system:

$$\begin{aligned}\dot{x}_i(t) &= -c_i(x_i(t)) + \sum_{j=1}^{m} w_{ij} h_j(y_j(t-\tau_{ij}))\\ \dot{y}_j(t) &= -d_j(y_j(t)) + \sum_{i=1}^{n} v_{ji} s_i(x_i(t-\sigma_{ji}))\end{aligned} \tag{15}$$

where

$$\begin{aligned}c_i(x_i(t)) &= p_i(x_i(t)+u_i^*) - p_i(u_i^*),\\ d_j(y_j(t)) &= q_j(y_j(t)+z_j^*) - q_j(z_j^*)\\ h_j(y_j(t)) &= f_j(y_j(t)+z_j^*) - f_j(z_j^*)\\ s_i(x_i(t)) &= g_i(x_i(t)+u_i^*) - g_i(u_i^*)\end{aligned}$$

and

$$\begin{aligned}x_i(t) &= \phi_i(t) = \phi_i^*(t) - u_i^*, && -\tau \le t \le 0\\ y_j(t) &= \psi_j(t) = \psi_j^*(t) - z_j^*, && -\sigma \le t \le 0.\end{aligned}$$

Since the transformation is a translation, we only need to prove that the origin (0, 0) of the transformed system is globally exponentially stable. We consider the following Lyapunov function:

$$V(x(t), y(t)) = e^{\theta t} V_1(x(t), y(t))$$

where

$$\begin{aligned}V_1(x(t), y(t)) = &\sum_{i=1}^{n} m\lambda_i a_i^{r-1}|x_i(t)|^r + \sum_{j=1}^{m} n\mu_j b_j^{r-1}|y_j(t)|^r\\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r \int_{t-\tau_{ij}}^{t} |h_j(y_j(s))|^r ds\\ &+ \sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r \int_{t-\sigma_{ji}}^{t} |s_i(x_i(s))|^r ds\end{aligned} \tag{16}$$

where the positive real number θ will be determined later.

The upper right Dini-derivative $D^+V(x(t), y(t))$ along the trajectories of the BAM NN given in (15) are given by

$$D^+V = \theta e^{\theta t} V_1(x(t), y(t)) + e^{\theta t} D^+V_1(x(t), y(t)). \tag{17}$$

Now

$$\begin{aligned}&D^+V_1(x(t), y(t))\\ &\quad\le r\sum_{i=1}^{n} m\lambda_i a_i^{r-1}|x_i(t)|^{r-1}\\ &\qquad\times\left[-a_i|x_i(t)| + \sum_{j=1}^{m}|w_{ij}||h_j(y_j(t-\tau_{ij}))|\right]\\ &\qquad + r\sum_{j=1}^{m} n\mu_j b_j^{r-1}|y_j(t)|^{r-1}\\ &\qquad\times\left[-b_j|y_j(t)| + \sum_{i=1}^{n}|v_{ji}||s_i(x_i(t-\sigma_{ji}))|\right]\\ &\qquad + \sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r|h_j(y_j(t))|^r\\ &\qquad - \sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r|h_j(y_j(t-\tau_{ij}))|^r\\ &\qquad + \sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r|s_i(x_i(t))|^r\\ &\qquad - \sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r|s_i(x_i(t-\sigma_{ji}))|^r\\ &\quad\le \sum_{i=1}^{n}\left\{-rm\lambda_i a_i^r|x_i(t)|^r\right.\\ &\qquad\quad + \sum_{j=1}^{m} r\lambda_i\left[\frac{r-1}{r}\left(a_i^{r-1}|x_i(t)|^{r-1}\right)^{r/r-1}\right.\\ &\qquad\qquad\left.\left. + \frac{1}{r}(m|w_{ij}||h_j(y_j(t-\tau_{ij}))|)^r\right]\right\}\\ &\qquad + \sum_{j=1}^{m}\left\{-rn\mu_j b_j^r|y_j(t)|^r\right.\\ &\qquad\quad + \sum_{i=1}^{n} r\mu_j\left[\frac{r-1}{r}\left(b_j^{r-1}|y_j(t)|^{r-1}\right)^{r/r-1}\right.\\ &\qquad\qquad\left.\left. + \frac{1}{r}(n|v_{ji}||s_i(x_i(t-\sigma_{ji}))|)^r\right]\right\}\\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r|h_j(y_j(t))|^r\\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r|h_j(y_j(t-\tau_{ij}))|^r\\ &\quad + \sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r|s_i(x_i(t))|^r\\ &\quad - \sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r|s_i(x_i(t-\sigma_{ji}))|^r\end{aligned}$$

so

$$\begin{aligned}&D^+V_1(x(t), y(t))\\ &\quad\le \sum_{i=1}^{n}\left\{-m\lambda_i a_i^r|x_i(t)|^r + \sum_{j=1}^{m}\lambda_i m^r|w_{ij}|^r\beta_j^r|y_j(t)|^r\right\}\\ &\qquad + \sum_{j=1}^{m}\left\{-n\mu_j b_j^r|y_j(t)|^r + \sum_{i=1}^{n}\mu_j n^r|v_{ji}|^r\alpha_i^r|x_i(t)|^r\right\}\\ &\quad = -\sum_{i=1}^{n} n^r\alpha_i^r\delta_i|x_i(t)|^r - \sum_{j=1}^{m} m^r\beta_j^r\Omega_j|y_j(t)|^r.\end{aligned} \tag{18}$$

Hence

$$\begin{aligned} &D^+V(x(t),y(t)) \\ &\quad\le e^{\theta t}\sum_{i=1}^{n}\left[\theta m\lambda_i a_i^{r-1} - n^r\alpha_i^r\delta_i\right]|x_i(t)|^r \\ &\qquad + e^{\theta t}\sum_{j=1}^{m}\left[\theta n\mu_j b_j^{r-1} - m^r\beta_j^r\Omega_j\right]|y_j(t)|^r \\ &\qquad + \theta e^{\theta t}\sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r\int_{t-\tau_{ij}}^{t}|h_j(y_j(s))|^r ds \\ &\qquad + \theta e^{\theta t}\sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r\int_{t-\sigma_{ji}}^{t}|s_i(x_i(s))|^r ds. \quad (19) \end{aligned}$$

Denote

$$\begin{aligned} G_i(\theta) &= \theta m\lambda_i a_i^{r-1} - n^r\alpha_i^r\delta_i + \theta\sum_{j=1}^{m} n^r\mu_j|v_{ji}|^r\alpha_i^r\sigma_{ji}e^{\sigma_{ji}\theta} \\ F_j(\theta) &= \theta n\mu_j b_j^{r-1} - m^r\beta_j^r\Omega_j + \theta\sum_{i=1}^{n} m^r\lambda_i|w_{ij}|^r\beta_j^r\tau_{ij}e^{\tau_{ij}\theta}. \end{aligned}$$

Noting that

$$\begin{aligned} |h_j(y_j(t))| &\le \beta_j|y_j(t)| \\ |s_i(x_i(t))| &\le \alpha_i|x_i(t)| \end{aligned}$$

by Lemma 7, we have

$$\begin{aligned} &V(x(t),y(t)) - V(x(0),y(0)) \\ &\quad\le \sum_{i=1}^{n}\left[\theta m\lambda_i a_i^{r-1} - n^r\alpha_i^r\delta_i\right]\int_0^t e^{\theta s}|x_i(s)|^r ds \\ &\qquad + \sum_{j=1}^{m}\left[\theta n\mu_j b_j^{r-1} - m^r\beta_j^r\Omega_j\right]\int_0^t e^{\theta s}|y_j(s)|^r ds \\ &\qquad + \theta\sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r\int_0^t e^{\theta s}ds\int_{s-\tau_{ij}}^{s}|h_j(y_j(v))|^r dv \\ &\qquad + \theta\sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r\int_0^t e^{\theta s}ds\int_{s-\sigma_{ji}}^{s}|s_i(x_i(v))|^r dv \\ &\quad\le \sum_{i=1}^{n}\left[\theta m\lambda_i a_i^{r-1} - n^r\alpha_i^r\delta_i\right]\int_0^t e^{\theta s}|x_i(s)|^r ds \\ &\qquad + \sum_{j=1}^{m}\left[\theta n\mu_j b_j^{r-1} - m^r\beta_j^r\Omega_j\right]\int_0^t e^{\theta s}|y_j(s)|^r ds \\ &\qquad + \theta\sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r\int_{-\tau_{ij}}^{t}|h_j(y_j(v))|^r\tau_{ij}e^{\theta(v+\tau_{ij})}dv \\ &\qquad + \theta\sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r\int_{-\sigma_{ji}}^{t}|s_i(x_i(v))|^r\sigma_{ji}e^{\theta(v+\sigma_{ji})}dv \\ &\quad\le \sum_{i=1}^{n}G_i(\theta)\int_0^t e^{\theta s}|x_i(s)|^r ds + \sum_{j=1}^{m}F_j(\theta)\int_0^t e^{\theta s}|y_j(s)|^r ds \\ &\qquad + \theta\sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r\int_{-\tau_{ij}}^{0}\beta_j^r|y_j(v)|^r\tau_{ij}e^{\theta(v+\tau_{ij})}dv \\ &\qquad + \theta\sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r\int_{-\sigma_{ji}}^{0}\alpha_i^r|x_i(v)|^r\sigma_{ji}e^{\theta(v+\sigma_{ji})}dv. \end{aligned}$$

(20)

Since $\Omega_j > 0$ and $\delta_i > 0$, $F_j(0) < 0$ and $G_i(0) < 0$. Since $F_j(\theta)$ and $G_i(\theta)$ are monotonic increasing continuous functions of θ in $[0,+\infty)$ and

$$\lim_{\theta\to+\infty}F_j(\theta) = \lim_{\theta\to+\infty}G_i(\theta) = +\infty$$

using the intermediate value theorem of continuous functions, there exist positive values $\gamma_i > 0$, $i = 1,\dots,n$, and $\gamma_{n+j} > 0$, $j = 1,\dots,m$, such that $G_i(\gamma_i) = 0$, $i = 1,\dots,n$, and $F_j(\gamma_{n+j}) = 0$, $j = 1,\dots,m$. Let

$$\theta_1 = \min_{1\le k\le n+m}\gamma_k.$$

Since each $\gamma_k > 0$, then $\theta_1 > 0$. Noting that $\theta_1 \le \gamma_k$ for $k = 1,\dots,n+m$, we have that $G_i(\theta_1) \le 0$, $i = 1,\dots,n$, and $F_j(\theta_1) \le 0$, $j = 1,\dots,m$. Setting θ to θ_1 in (20) gives

$$\begin{aligned} &e^{\theta_1 t}V_1(x(t),y(t)) - V_1(x(0),y(0)) \\ &\quad\le \theta_1\sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r\beta_j^r\|\psi\|^r\tau_{ij}^2e^{\theta_1\tau_{ij}} \\ &\qquad + \theta_1\sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r\alpha_i^r\|\phi\|^r\sigma_{ji}^2e^{\theta_1\sigma_{ji}}. \quad (21) \end{aligned}$$

Setting $t = 0$ in (16) gives

$$\begin{aligned} V_1(x(0),y(0)) \le{}& \sum_{i=1}^{n} m\lambda_i a_i^{r-1}\|\phi\|^r + \sum_{j=1}^{m} n\mu_j b_j^{r-1}\|\psi\|^r \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m} m^r\lambda_i|w_{ij}|^r\beta_j^r\|\psi\|^r\tau_{ij} \\ &+ \sum_{j=1}^{m}\sum_{i=1}^{n} n^r\mu_j|v_{ji}|^r\alpha_i^r\|\phi\|^r\sigma_{ji}. \quad (22) \end{aligned}$$

Combining (21) and (22) gives

$$e^{\theta_1 t}V_1(x(t),y(t)) \le \pi(\|\psi\|^r + \|\phi\|^r) \quad (23)$$

where $\pi = \max\{\pi_1,\pi_2\}$ and π_1 and π_2 are, respectively, given by

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\mu_j b_j^{r-1} + m^r\lambda_i|w_{ij}|^r\beta_j^r\tau_{ij}\left(1 + \theta_1\tau_{ij}e^{\theta_1\tau_{ij}}\right)\right] \\ &\sum_{j=1}^{m}\sum_{i=1}^{n}\left[\lambda_i a_i^{r-1} + n^r\mu_j|v_{ji}|^r\alpha_i^r\sigma_{ji}\left(1 + \theta_1\sigma_{ji}e^{\theta_1\sigma_{ji}}\right)\right]. \end{aligned}$$

Let

$$q = \min\left\{\min_{1\le i\le n} m\lambda_i a_i^{r-1},\ \min_{1\le j\le m} n\mu_j b_j^{r-1}\right\}.$$

Clearly, $q > 0$. From (16), it is clear that

$$q\left[\sum_{i=1}^{n}|x_i(t)|^r+\sum_{j=1}^{m}|y_j(t)|^r\right]e^{\theta_1 t}\le e^{\theta_1 t}V_1(x(t),y(t)) \le \pi(\|\psi\|^r+\|\phi\|^r). \quad (24)$$

It follows that

$$\sum_{i=1}^{n}|x_i(t)|^r+\sum_{j=1}^{m}|y_j(t)|^r\le\frac{\pi}{q}e^{-\theta_1 t}(\|\psi\|+\|\phi\|)^r$$

for any $t \ge 0$, where $\pi/q \ge 1$ is a constant. This implies that the solution of (15), at the origin, is globally exponentially stable.

This completes the proof of Theorem 1. ■

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