Existence and Uniqueness of Periodic Solutions of a Liénard Equation with Delay

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Abstract

Using inequality techniques and coincidence degree theory, new results are provided concerning the existence and uniqueness of T-periodic solutions for a Liénard equations with delay. An illustrative example is provided to demonstrate that the results in this paper hold under weaker conditions than existing results, and are more effective.

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1 Introduction

There has been a great deal of work on Liénard equations, which have been used to describe fluid-mechanical and nonlinear elastic mechanical phenomena. For example, in [2, 3, 4, 8] and [13], time maps and phase plane analysis were used to examine the existence of periodic solutions to Liénard equations, and several sufficient conditions for this existence were established. Recently, Liu and Huang [9] discussed the existence and uniqueness of periodic solutions for a Liénard equation with delay, of form

$$x''(t) + f(x(t))x'(t) + g(t, x(t - \tau(t))) = p(t) \tag{1}$$

where $f, \tau, p : R \to R$ and $g : R \times R \to R$ are continuous functions, τ and p are T-periodic (i.e. periodic with period T), g is T-periodic in its first argument, and $T > 0$.

In recent years, periodic solutions to Eq.(1) have been extensively studied in the literature (see, for example, [1] and [5, 7, 10, 11, 12, 14]). However, to the best of our knowledge, most authors have only considered the *existence* of periodic solutions, and few results exist concerning both existence and *uniqueness* of periodic solutions to Eq.(1). Liu and Huang [9] provide a sufficient condition for such existence and uniqueness, but their result leaves space for improvement. In their work, they required that the following constraint should be imposed on the Liénard equation:

$$C_1 D \frac{T}{2\pi} + C_2 \frac{T}{2\pi} + \frac{bT^2}{2\pi} < 1.$$

Here, C_1, C_2 and b are nonnegative constants determined by

$$|f(x_1) - f(x_2)| \leq C_1 |x_1 - x_2|, \qquad |f(x)| \leq C_2, \qquad C_2 \frac{T}{2\pi} + b\frac{T^2}{4\pi} < 1,$$

$$|g(t, x_1) - g(t, x_2)| \leq b|x_1 - x_2|, \qquad \forall t, x_1, x_2, x \in R;$$

and d is a positive constant such that one of the following conditions holds:

$$x(g(t, x) - p(t)) < 0,\ \forall t \in R, |x| \geq d \qquad \text{or} \qquad x(g(t, x) - p(t)) > 0,\ \forall t \in R, |x| \geq d.$$

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Furthermore,

$$D = \frac{[bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]T}{1 - (C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi})}.$$

Upon examining their proof of Lemma 2.5 and Theorem 3.1 in [9], we have found certain errors. The corrected version of this constraint should read

$$C_1 D\frac{\sqrt{T^3}}{2\pi} + C_2\frac{T}{2\pi} + \frac{bT^2}{2\pi} < 1.$$

In this paper, we reconsider periodic solutions of a Liénard equation with delay as given in Eq.(1). The main purpose of this paper is to establish a new sufficient condition for the existence and uniqueness of T-periodic solutions of Eq.(1). Using inequality techniques, we obtain sharp *a priori* estimates for a periodic solution to Eq.(1). Furthermore, by using improved estimates for $|x|_\infty, |x'|_\infty$ and $|x'|_2$ and coincidence degree theory, both the existence and uniqueness of T-periodic solutions of Eq.(1) under this sufficient condition are proved. This sufficient condition improves upon the main result obtained in [9], as we demonstrate using an illustrative example.

For simplicity, throughout this paper, we adopt the following notation:

$$|x|_k = (\int_0^T |x(t)|^k dt)^{1/k}$$

and

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|.$$

In order to use Mawhin's continuation theorem to study the existence of periodic solution of Eq.(1), we introduce the following spaces and operators. Let

$$X = \{x | x \in C^1(R,R),\ x(t+T) = x(t),\ \forall t \in R\}$$

and

$$Y = \{x | x \in C(R,R),\ x(t+T) = x(t),\ \forall t \in R\}$$

be two Banach spaces with norms

$$\|x\|_X = \max\{|x|_\infty, |x'|_\infty\} \quad \text{and} \quad \|x\|_Y = |x|_\infty.$$

Let $D(L) = \{x | x \in X,\ x'' \in C(R,R)\}$. Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$Lx = x''. \tag{2}$$

We also define a nonlinear operator $N : X \to Y$ by setting

$$Nx = -f(x(t))x'(t) - g(t, x(t-\tau(t))) + p(t). \tag{3}$$

Obviously, Ker$L = R$, and Im$L = \{x | x \in Y, \int_0^T x(s)ds = 0\}$. Thus the operator L is a Fredholm operator with index zero. Define the continuous projector $P : X \to$ Ker$L$ and the averaging projector $Q : Y \to Y$ by setting $Px(t) = x(0) = x(T)$ and $Qx(t) = \frac{1}{T}\int_0^T x(s)ds$. Hence, Im$P =$ Ker$L$ and Ker$Q =$ ImL. Denoting by L_P^{-1} : Im$L \to D(L) \cap$ Ker$P$ the inverse of $L|_{D(L)\cap \mathrm{Ker}P}$, we have

$$L_P^{-1}y(t) = -\frac{t}{T}\int_0^T (t-s)y(s)ds + \int_0^t (t-s)y(s)ds. \tag{4}$$

It is convenient to introduce the following assumption
(A_0): there exist nonnegative constants C_1 and C_2 such that

$$|f(x_1) - f(x_2)| \le C_1|x_1 - x_2|, \quad |f(x)| \le C_2$$

for all $x_1, x_2, x \in R$.

2 Preliminaries

In view of Eqs.(2) and (3), the operator equation $Lx = \lambda Nx$ is equivalent to the following:

$$x''(t) + \lambda[f(x(t))x'(t) + g(t, x(t - \tau(t)))] = \lambda p(t) \quad (5)$$

where $\lambda \in (0, 1)$.

For convenience of use, we introduce the Continuation Theorem [5] as follows.

Lemma 1. *Let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on the closure $\bar{\Omega}$ of Ω, where Ω is an open bounded subset of X. Moreover, assume that each of the following conditions is satisfied:*

1. $Lx \neq \lambda Nx, \ \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;
2. $Nx \notin \mathrm{Im}L, \forall x \in \partial\Omega \cap \mathrm{Ker}L$;
3. *The Brouwer degree* $\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$.

Then equation $Lx = Nx$ has at least one solution on $\bar{\Omega}$.

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2. *See [6, 14]. If $x \in C^2(R, R)$ with $x(t + T) = x(t)$, then*

$$|x'(t)|_2^2 \leq (\frac{T}{2\pi})^2 |x''(t)|_2^2. \quad (6)$$

Lemma 3. *Suppose that there exists a constant $d > 0$ such that one of the following conditions holds:*

(A_1): $x(g(t, x) - p(t)) < 0, \ \forall t \in R, |x| \geq d$;

(A_2): $x(g(t, x) - p(t)) > 0, \ \forall t \in R, |x| \geq d$.

If $x(t)$ is a T-periodic solution of (5), then

$$|x|_\infty \leq d + \frac{\sqrt{T}}{2}|x'|_2. \quad (7)$$

Proof. Let $x(t)$ be a T-periodic solution of Eq.(5). Then, integrating Eq.(5) from 0 to T, we have

$$\int_0^T [g(t, x(t - \tau(t))) - p(t)]dt = 0. \quad (8)$$

This implies that there exists $\xi \in [0, T]$ such that

$$g(\xi, x(\xi - \tau(\xi))) - p(\xi) = 0. \quad (9)$$

Taking this together with (A_1) or (A_2) as appropriate, we have

$$|x(\xi - \tau(\xi))| < d. \quad (10)$$

Let $\xi - \tau(\xi) = mT + t_0$, where $t_0 \in [0, T]$ and m is an integer. Then,

$$\begin{aligned} |x(t)| &= |x(t_0) + \int_{t_0}^t x'(s)ds| \\ &= |x(mT + t_0) + \int_{t_0}^t x'(s)ds| \\ &\leq |x(\xi - \tau(\xi))| + |\int_{t_0}^t x'(s)ds| \\ &< d + \int_{t_0}^t |x'(s)|ds \end{aligned} \quad (11)$$

where $t \in [t_0, t_0 + T]$.

Since $x(t)$ is the T-periodic solution, for $t \in [t_0, t_0 + T]$,

$$\begin{aligned} |x(t)| &= |x(t_0 + T) + \int_{t_0+T}^{t} x'(s)ds| \\ &= |x(t_0 + T) - \int_{t}^{t_0+T} x'(s)ds| \\ &\leq |x(t_0 + T)| + |\int_{t}^{t_0+T} x'(s)ds| \\ &\leq |x(t_0 + T)| + \int_{t}^{t_0+T} |x'(s)|ds \\ &\leq d + \int_{t}^{t_0+T} |x'(s)|ds. \end{aligned} \tag{12}$$

Combining Eqs.(11) and (12) gives

$$\begin{aligned} |x(t)| &\leq d + \frac{1}{2}\int_{t_0}^{t_0+T} |x'(s)|ds \\ &\leq d + \frac{1}{2}\int_{0}^{T} |x'(s)|ds. \end{aligned} \tag{13}$$

Using the Schwartz inequality yields

$$\begin{aligned} |x(t)| &\leq d + \frac{1}{2}\sqrt{T}(\int_{0}^{T} |x'(s)|^2 ds)^{1/2} \\ &= d + \frac{1}{2}\sqrt{T}|x'|_2. \end{aligned} \tag{14}$$

Therefore,

$$|x|_\infty = \max_{t\in[0,T]} |x(t)| \leq d + \frac{1}{2}\sqrt{T}|x'|_2. \tag{15}$$

This completes the proof of Lemma 3. □

Lemma 4. *Let (A_0) and either (A_1) or (A_2) hold. Assume that the following condition is satisfied:*

(A_3)*: There exists a nonnegative constant b such that*

$$C_2\frac{T}{2\pi} + b\frac{T^2}{4\pi} < 1, \quad \text{and} \quad |g(t, x_1) - g(t, x_2)| \leq b|x_1 - x_2|, \quad \forall t, x_1, x_2 \in R.$$

If $x(t)$ is a T-periodic solution of Eq.(1), then

$$|x'|_2 \leq D_1 \tag{16}$$

where

$$D_1 = \frac{[bd + \max\{|g(t, 0)| : 0 \leq t \leq T\} + |p|_\infty]T}{2(1 - C_2\frac{T}{2\pi} - b\frac{T^2}{4\pi})}.$$

Proof. Let $x(t)$ be a T-periodic solution of Eq.(1). From (A_1) or (A_2), we can easily show that Inequality (7) also holds. Multiplying Eq.(1) by $x''(t)$ and then integrating from 0 to T, in view of Inequalities (6) and (7), (A_3) and the Schwartz inequality, we have

$$|x''|_2^2 = -\int_0^T f(x(t))x'(t)x''(t)dt - \int_0^T g(t, x(t - \tau(t)))x''(t)dt + \int_0^T p(t)x''(t)dt$$

$$
\begin{aligned}
&\leq \int_0^T |f(x(t))||x'(t)||x''(t)|dt + \int_0^T |p(t)||x''(t)|dt \\
&\quad + \int_0^T [|g(t,x(t-\tau(t))) - g(t,0)| + |g(t,0)|]|x''(t)|dt \\
&\leq C_2 \int_0^T |x'(t)||x''(t)|dt + |p|_\infty \int_0^T |x''(t)|dt \\
&\quad + b\int_0^T |x(t-\tau(t))||x''(t)|dt + \int_0^T |g(t,0)||x''(t)|dt \\
&\leq C_2|x'|_2|x''|_2 + |p|_\infty \sqrt{T}|x''|_2 \\
&\quad + b\int_0^T |x(t-\tau(t))||x''(t)|dt + \int_0^T |g(t,0)||x''(t)|dt \\
&\leq C_2\frac{T}{2\pi}|x''|_2^2 + |p|_\infty \sqrt{T}|x''|_2 \\
&\quad + b|x|_\infty \sqrt{T}|x''|_2 + \max\{|g(t,0)| : 0 \leq t \leq T\}\sqrt{T}|x''|_2 \\
&\leq C_2\frac{T}{2\pi}|x''|_2^2 + |p|_\infty \sqrt{T}|x''|_2 \\
&\quad + b(d + \frac{\sqrt{T}}{2}|x'|_2)\sqrt{T}|x''|_2 + \max\{|g(t,0)| : 0 \leq t \leq T\}\sqrt{T}|x''|_2 \\
&\leq C_2\frac{T}{2\pi}|x''|_2^2 + |p|_\infty \sqrt{T}|x''|_2 \\
&\quad + b(d + \frac{\sqrt{T}}{2}\frac{T}{2\pi}|x''|_2)\sqrt{T}|x''|_2 + \max\{|g(t,0)| : 0 \leq t \leq T\}\sqrt{T}|x''|_2 \\
&\leq [C_2\frac{T}{2\pi} + b\frac{T^2}{4\pi}]|x''|_2^2 \\
&\quad + [bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]\sqrt{T}|x''|_2. \qquad (17)
\end{aligned}
$$

Thus,

$$
[1 - C_2\frac{T}{2\pi} - b\frac{T^2}{4\pi}]|x''|_2 \leq [bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]\sqrt{T}, \qquad (18)
$$

so

$$
|x''|_2 \leq \frac{[bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]\sqrt{T}}{1 - C_2\frac{T}{2\pi} - b\frac{T^2}{4\pi}}. \qquad (19)
$$

Since $x(0) = x(T)$, there exists a constant $\zeta \in [0, T]$ such that $x'(\zeta) = 0$, and

$$
\begin{aligned}
|x'(t)| &= |x'(\zeta) + \int_\zeta^t x''(s)ds| \\
&\leq |x'(\zeta)| + \int_\zeta^t |x''(s)|ds \\
&= \int_\zeta^t |x''(s)|ds \qquad (20)
\end{aligned}
$$

where $t \in [\zeta, T + \zeta]$. Again,

$$
\begin{aligned}
|x'(t)| &= |x'(\zeta + T) + \int_{\zeta+T}^t x''(s)ds| \\
&= |x'(\zeta + T) - \int_t^{\zeta+T} x''(s)ds| \\
&\leq |x'(\zeta + T)| + |\int_t^{\zeta+T} x''(s)ds|
\end{aligned}
$$

$$\begin{aligned} &\leq |x'(\zeta+T)| + \int_t^{\zeta+T} |x''(s)|ds \\ &= |x'(\zeta)| + \int_t^{\zeta+T} |x''(s)|ds \\ &= \int_t^{\zeta+T} |x''(s)|ds \end{aligned} \tag{21}$$

where $t \in [0, T]$. Inequalities (20) and (21) imply that

$$\begin{aligned} 2|x'(t)| &\leq \int_\zeta^t |x''(s)|ds + \int_t^{\zeta+T} |x''(s)|ds \\ &= \int_\zeta^{\zeta+T} |x''(s)|ds \\ &= \int_0^T |x''(s)|ds \\ &\leq \sqrt{T}|x''|_2 \end{aligned} \tag{22}$$

where $t \in [0, T]$.

Obviously,

$$|x'(t)| \leq \frac{1}{2}\sqrt{T}|x''|_2 \quad \text{for all} \quad t \in [0, T], \tag{23}$$

so

$$|x'|_\infty \leq \frac{1}{2}\sqrt{T}|x''|_2. \tag{24}$$

From Inequalities (19) and (24), we have

$$|x'|_\infty \leq \frac{[bd + \max\{|g(t, 0)| : 0 \leq t \leq T\} + |p|_\infty]T}{2(1 - C_2\frac{T}{2\pi} - b\frac{T^2}{4\pi})}. \tag{25}$$

This completes the proof of Lemma 4. □

Lemma 5. *Let (A_1) or (A_2) hold. Assume that the following condition is satisfied:*

(A_4) *Suppose that (A_0) holds, $g(t, x)$ is a strictly monotone function in x, and there exists a nonnegative constant b such that*

$$C_1 D_1 \frac{\sqrt{T^3}}{4\pi} + C_2\frac{T}{2\pi} + \frac{bT^2}{4\pi} < 1, \quad \text{and} \quad |g(t, x_1) - g(t, x_2)| \leq b|x_1 - x_2|$$

where C_1, C_2, D_1 are defined as before.

Then Eq.(1) has at most one T-periodic solution.

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of Eq.(1). Then

$$(x_1(t) - x_2(t))'' + (f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t)) + (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t)))) = 0. \tag{26}$$

Set $Z(t) = x_1(t) - x_2(t)$. Then, from Eq.(26), we obtain

$$Z''(t) + (f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t)) + (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t)))) = 0. \tag{27}$$

Since $x_1(t)$ and $x_2(t)$ are T-periodic, integrating Eq.(27) from 0 to T, we obtain

$$\int_0^T (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t))))dt = 0. \tag{28}$$

Using the integral mean value theorem, it follows that there exists a constant $\gamma \in [0, T]$ such that

$$g(\gamma, x_1(\gamma - \tau(\gamma))) - g(\gamma, x_2(\gamma - \tau(\gamma))) = 0. \tag{29}$$

Let $\gamma - \tau(\gamma) = nT + \bar{\gamma}$, where $\bar{\gamma} \in [0, T]$ and n is an integer. Then, Eq.(29), together with (A_4), implies that there exists a constant $\bar{\gamma} \in [0, T]$ such that

$$Z(\bar{\gamma}) = x_1(\bar{\gamma}) - x_2(\bar{\gamma}) = x_1(\gamma - \tau(\gamma)) - x_2(\gamma - \tau(\gamma)) = 0. \tag{30}$$

Thus,

$$\begin{aligned} |Z(t)| &= |Z(\bar{\gamma}) + \int_{\bar{\gamma}}^{t} Z'(s)ds| \\ &= |\int_{\bar{\gamma}}^{t} Z'(s)ds| \\ &\leq \int_{\bar{\gamma}}^{t} |Z'(s)|ds. \end{aligned} \tag{31}$$

Again

$$\begin{aligned} |Z(t)| &= |Z(\bar{\gamma} + T) + \int_{\bar{\gamma}+T}^{t} Z'(s)ds| \\ &= |Z(\bar{\gamma} + T) - \int_{t}^{\bar{\gamma}+T} Z'(s)ds| \\ &= |-\int_{t}^{\bar{\gamma}+T} Z'(s)ds| \\ &\leq \int_{t}^{\bar{\gamma}+T} |Z'(s)|ds. \end{aligned} \tag{32}$$

Hence

$$\begin{aligned} 2|Z(t)| &\leq \int_{\bar{\gamma}}^{t} |Z'(s)|ds + \int_{t}^{\bar{\gamma}+T} |Z'(s)|ds \\ &\leq \int_{\bar{\gamma}}^{\bar{\gamma}+T} |Z'(s)|ds \\ &= \int_{0}^{T} |Z'(s)|ds \\ &\leq \sqrt{T}(\int_{0}^{T} |Z'(s)|^2 ds)^{1/2} \\ &= \sqrt{T}|Z'|_2. \end{aligned} \tag{33}$$

Hence

$$|Z|_\infty \leq \frac{1}{2}\sqrt{T}|Z'|_2. \tag{34}$$

Multiplying Eq.(27) by $Z''(t)$, then integrating from 0 to T, from Inequalities (6) and (34), and the Schwartz inequality, we get

$$\begin{aligned} |Z''|_2^2 &= -\int_{0}^{T} (f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t))Z''(t)dt \\ &\quad - \int_{0}^{T} (g(t, x_1(t - \tau(t))) - g(t, x_2(t - \tau(t))))Z''(t)dt \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T |f(x_1(t))||x_1'(t)-x_2'(t)||Z''(t)|dt \\
&\quad +\int_0^T |f(x_1(t))-f(x_2(t))||x_2'(t)||Z''(t)|dt \\
&\quad +b\int_0^T |x_1(t-\tau(t))-x_2(t-\tau(t))||Z''(t)|dt \\
&\leq \int_0^T C_2|Z'(t)||Z''(t)|dt \\
&\quad +\int_0^T C_1|Z(t)||x_2'(t)||Z''(t)|dt \\
&\quad +b\int_0^T |Z(t-\tau(t))||Z''(t)|dt \\
&\leq C_2(\int_0^T |Z'(t)|^2dt)^{1/2}(\int_0^T |Z''(t)|^2dt)^{1/2} \\
&\quad +C_1|Z|_\infty\int_0^T |x_2'(t)||Z''(t)|dt \\
&\quad +b|Z|_\infty\int_0^T |Z''(t)|dt \\
&\leq C_2|Z'|_2|Z''|_2+C_1|Z|_\infty|x_2'|_2|Z''|_2+b|Z|_\infty\sqrt{T}|Z''|_2 \\
&\leq [C_2+\frac{1}{2}C_1\sqrt{T}D_1+\frac{1}{2}bT]|Z'|_2|Z''|_2 \\
&\leq [\frac{1}{2}C_1\sqrt{T}D_1+C_2+\frac{1}{2}bT]\frac{T}{2\pi}|Z''|_2^2 \\
&= [C_1D_1\frac{\sqrt{T^3}}{4\pi}+C_2\frac{T}{2\pi}+\frac{bT^2}{4\pi}]|Z''|_2^2. \qquad (35)
\end{aligned}
$$

Since $Z(t)$, $Z'(t)$ and $Z''(t)$ are T-periodic and continuous functions, in view of (A_4) and Inequalities (6), (30) and (35), we have

$$Z(t)=Z'(t)=Z''(t)\equiv 0, \quad \forall t\in R.$$

Thus,

$$x_1(t)\equiv x_2(t), \quad \forall t\in R.$$

Therefore, Eq.(1) has at most one T-periodic solution. □

3 Main result

Theorem 1. *Let (A_1) or (A_2) hold. Assume that condition (A_4) is satisfied. Then Eq. (1) has a unique T-periodic solution.*

Proof. Lemma 5 states that Eq. (1) has at most one T-periodic solution. Thus, to prove Theorem 1, it suffices to show that Eq.(1) has at least one T-periodic solution. To do this, we apply Lemma 1. Firstly, we claim that the set of all possible T-periodic solutions of Eq.(5) is bounded. Let $x(t)$ be a T-periodic solution of Eq.(5). Multiplying Eq.(5) by $x''(t)$, then integrating from 0 to T, and using Lemmas 2 and 3, Assumption (A_4) and the Schwartz inequality, we have

$$
\begin{aligned}
|x''|_2^2 &= -\lambda\int_0^T f(x(t))x'(t)x''(t)dt-\lambda\int_0^T g(t,x(t-\tau(t)))x''(t)dt+\lambda\int_0^T p(t)x''(t)dt \\
&\leq \int_0^T |f(x(t))||x'(t)||x''(t)|dt+\int_0^T |p(t)||x''(t)|dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T [|g(t,x(t-\tau(t)))-g(t,0)|+|g(t,0)|]|x''(t)|dt \\
\leq\ & C_2\int_0^T |x'(t)||x''(t)|dt+|p|_\infty\int_0^T |x''(t)|dt \\
&+b\int_0^T |x(t-\tau(t))||x''(t)|dt+\int_0^T |g(t,0)||x''(t)|dt \\
\leq\ & C_2|x'|_2|x''|_2+|p|_\infty\sqrt{T}|x''|_2 \\
&+b|x|_\infty\sqrt{T}|x''|_2+\max\{|g(t,0)|:0\leq t\leq T\}\sqrt{T}|x''|_2 \\
\leq\ & C_2\frac{T}{2\pi}|x''|_2^2+|p|_\infty\sqrt{T}|x''|_2 \\
&+b(d+\frac{\sqrt{T}}{2}|x'|_2)\sqrt{T}|x''|_2+\max\{|g(t,0)|:0\leq t\leq T\}\sqrt{T}|x''|_2 \\
\leq\ & C_2\frac{T}{2\pi}|x''|_2^2+|p|_\infty\sqrt{T}|x''|_2 \\
&+b(d+\frac{\sqrt{T}}{2}\frac{T}{2\pi}|x''|_2)\sqrt{T}|x''|_2+\max\{|g(t,0)|:0\leq t\leq T\}\sqrt{T}|x''|_2 \\
\leq\ & [C_2\frac{T}{2\pi}+b\frac{T^2}{4\pi}]|x''|_2^2 \\
&+[bd+\max\{|g(t,0)|:0\leq t\leq T\}+|p|_\infty]\sqrt{T}|x''|_2 \qquad (36)
\end{aligned}
$$

which, together with (A_4), implies that there exist positive constants D_3 and D_4 such that

$$|x''|_2<D_3,\quad |x'|_2<D_4,\quad |x|_\infty<D_4. \qquad (37)$$

Since $x(0)=x(T)$, there exists a constant $\xi\in[0,T]$ such that

$$x'(\xi)=0$$

and

$$
\begin{aligned}
|x'(t)| &= |x'(\xi)+\int_\xi^t x''(s)ds| \\
&= |\int_\xi^t x''(s)ds| \\
&\leq \int_\xi^t |x''(s)|ds \qquad (38)
\end{aligned}
$$

where $t\in[\xi,T+\xi]$. Similarly,

$$
\begin{aligned}
|x'(t)| &= |x'(\xi+T)+\int_{\xi+T}^t x''(s)ds| \\
&= |x'(\xi+T)-\int_t^{\xi+T} x''(s)ds| \\
&= |\int_t^{\xi+T} x''(s)ds| \\
&\leq \int_t^{\xi+T} |x''(s)|ds
\end{aligned}
\qquad (39)
$$

where $t\in[\xi,T+\xi]$. So

$$|x'(t)| \leq \frac{1}{2}(\int_\xi^t |x''(s)|ds+\int_t^{\xi+T} |x''(s)|ds|)$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\xi}^{\xi+T}|x''(s)|ds \\
&= \frac{1}{2}\int_{0}^{T}|x''(s)|ds \\
&\leq \frac{1}{2}\sqrt{T}|x''|_2 \\
&\leq \frac{1}{2}\sqrt{T}D_3. \qquad (40)
\end{aligned}
$$

From Inequalities (37) and (40), there exists a positive constant $M_1 > \frac{1}{2}\sqrt{T}D_3 + D_4$ such that

$$\|x\|_X \leq |x|_\infty + |x'|_\infty < M_1.$$

If $x \in \Omega_1 = \{x|x \in \mathrm{Ker}L \cap X, \text{and}\, Nx \in \mathrm{Im}L\}$, then there exists a constant M_2 such that

$$x(t) \equiv M_2,$$

and

$$\int_0^T [g(t, M_2) - p(t)]dt = 0. \qquad (41)$$

Thus

$$\int_0^T M_2[g(t, M_2) - p(t)]dt = 0. \qquad (42)$$

Hence, $\forall x(t) \in \Omega_1$,

$$|x(t)| \equiv |M_2| < d. \qquad (43)$$

Let $M = M_1 + d + 1$. Set

$$\Omega = \{x|x \in X, |x|_\infty < M, |x'|_\infty < M\}.$$

It is easy to see from Eqs.(3) and (4) that N is L-compact on $\bar{\Omega}$. It follows from Eqs.(41) and (43), and the fact $M > \max\{M_1, d\}$, that conditions (1) and (2) in Lemma 1 hold.

Furthermore, suppose we define continuous functions $H_1(x, \mu)$ and $H_2(x, \mu)$ as

$$
\begin{aligned}
H_1(x, \mu) &= (1 - \mu)x - \mu\frac{1}{T}\int_0^T [g(t, x) - p(t)]dt; \mu \in [0, 1], \\
H_2(x, \mu) &= -(1 - \mu)x - \mu\frac{1}{T}\int_0^T [g(t, x) - p(t)]dt; \mu \in [0, 1]. \qquad (44)
\end{aligned}
$$

If (A_1) holds , then

$$xH_1(x, \mu) \neq 0 \qquad (45)$$

where $x \in \partial\Omega \cap \mathrm{Ker}L$. Hence, using the homotopy invariance theorem, we have

$$
\begin{aligned}
\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} &= \deg\{-\frac{1}{T}\int_0^T [g(t, x) - p(t)]dt, \Omega \cap \mathrm{Ker}L, 0\} \\
&= \deg\{x, \Omega \cap \mathrm{Ker}L, 0\} \\
&\neq 0. \qquad (46)
\end{aligned}
$$

If (A_2) holds , then

$$xH_2(x, \mu) \neq 0 \qquad (47)$$

where $x \in \partial\Omega \cap \mathrm{Ker}L$. Hence, using the homotopy invariance theorem, we have

$$
\begin{aligned}
\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} &= \deg\{-\frac{1}{T}\int_0^T [g(t, x) - p(t)]dt, \Omega \cap \mathrm{Ker}L, 0\} \\
&= \deg\{-x, \Omega \cap \mathrm{Ker}L, 0\} \\
&\neq 0. \qquad (48)
\end{aligned}
$$

From Lemma 1, we conclude that Theorem 1 is proved. □

4 Example

In this section, we provide an example to illustrate that the sufficient condition given in Theorem 1 in this paper is superior to the recently reported result in [9].

Example 1. *Let $g(t,x) = x/(6\pi)$ $\forall t, x \in R$. Consider the existence and uniqueness of a 2π-periodic solution to the following Liénard equation:*

$$x''(t) + \frac{3}{8}(\sin x(t))x'(t) + g(t, x(t - \sin^2 t)) = \frac{1}{6\pi}e^{\cos(t)-1}. \tag{49}$$

In this example, $d = 1, b = 1/(6\pi), C_1 = C_2 = 3/8, \tau(t) = \sin^2 t, T = 2\pi$ and $p(t) = \frac{1}{6\pi}e^{\cos(t)-1}$. It is obvious that assumptions (A_2) and (A_4) both hold. Using the method in Theorem 1 in [9], we find that

$$\begin{aligned} D &= \frac{[bd + \max\{|g(t,0) : 0 \le t \le T|\} + |p|_\infty]T}{1 - (C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi})} \\ &= \frac{[\frac{1}{6\pi} + \frac{1}{6\pi}] \cdot 2\pi}{1 - \frac{3}{8} - \frac{1}{3}} \\ &= 2.2857, \end{aligned}$$

$$C_1 D\frac{\sqrt{T^3}}{2\pi} + C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi} = 2.8569 > 1.$$

Since $2.8569 > 1$, *the condition in Theorem 1 in [9] is not satisfied and hence it cannot provide any result. Therefore, Theorem 1 in [9] fails, while, our criterion in Theorem 1 in this paper remains applicable, as we now show.*

In fact,

$$\begin{aligned} D_1 &= \frac{[bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]T}{2(1 - C_2\frac{T}{2\pi} - b\frac{T^2}{4\pi})} \\ &= \frac{[\frac{1}{6\pi} + \frac{1}{6\pi}]2\pi}{2 - \frac{3}{4} - \frac{1}{3}} \\ &= 0.7273 \end{aligned}$$

and

$$C_1 D_1\frac{\sqrt{T^3}}{4\pi} + C_2\frac{T}{2\pi} + b\frac{T^2}{4\pi} = 0.8835 < 1.$$

Thus, Theorem 1 here shows that Eq.(49) has a unique 2π-periodic solution.

This example demonstrates that the condition in our Theorem 1 is weaker than that in [9], and is able to demonstrate existence of unique solutions to certain Liénard equations which the latter cannot decide about. Therefore our results substantially improve the works in [9].

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